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SETS WITH NO SUBSETS OF HIGHER WEAK TRUTH-TABLE DEGREE

A b s t r a c t. We consider the weak truth-table reducibility \leq_{wtt} and we prove the existence of *wtt*-introimmune sets in Δ_2^0 . This closes the gap on the existence of arithmetical *r*-introimmune sets for all the known reducibilities \leq_r strictly contained in the Turing reducibility.

1. Introduction

The existence of sets without subsets of higher Turing degree was proved by Soare [11]. In terms of their complexity, we know by Jockusch [7] that they cannot be arithmetical, and later Simpson [10] even proved that they cannot be hyperarithmetical. A natural question is to consider reducibilities \leq_r that are strictly contained in the Turing reducibility \leq_T and to

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see if there are arithmetical sets without subsets of higher r -degree. The reader unfamiliar with these reducibilities can see e.g. [6, 8, 9, 12]. The approach of to consider such reducibilities \leq_r and to study the existence of arithmetical sets without subsets of higher r -degree was initiated in [5], in which r -*introimmune* sets have been introduced. An infinite set A of natural numbers is r -introimmune if for every subset B of A with $|A \setminus B| = \infty$ we have $A \not\leq_r B$. Some common reducibilities strictly contained in \leq_T studied in Computability Theory are the following, from the smallest to the largest: the *one-one* \leq_1 , the *many-one* \leq_m , the *truth-table* \leq_{tt} and the *weak truth-table* reducibility \leq_{wtt} . r -introimmune sets have no subsets of higher r -degree for all the reducibilities \leq_r of the list. In [5] it was proved the existence of arithmetical c -introimmune sets, where \leq_c is the conjunctive reducibility, a particular truth-table reducibility. More specifically, it was proved the existence of c -introimmune Δ_4^0 sets. This was improved by Ambos-Spies [1] by showing the existence of tt -introimmune Δ_2^0 sets. So, from Ambos-Spies' result we know that there are arithmetical r -introimmune sets for all the reducibilities \leq_r of the above list up to \leq_{tt} . In this paper we close the gap by considering the *weak truth-table* reducibility \leq_{wtt} , and we prove the existence of arithmetical wtt -introimmune sets, in particular wtt -introimmune Δ_2^0 sets. Since we currently do not know intermediate reducibilities between \leq_{wtt} and \leq_T , we deduce that for all the known reducibilities \leq_r strictly contained in \leq_T there are arithmetical r -introimmune sets.

2. Notation

Our notation is standard and we mainly refer to [9, 12]. Letter \mathbb{N} denotes the set of natural numbers. We identify each subset of \mathbb{N} with its characteristic function. Given any two sets $A, B \subseteq \mathbb{N}$, $A \setminus B$ denotes the set difference of A and B . We fix a computable permutation $\langle \cdot, \cdot \rangle : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$. A string is any function $\alpha : \{0, 1, \dots, n\} \rightarrow \{0, 1\}$, where $n \in \mathbb{N}$. \emptyset denotes the empty string. The length of a string α , in short $|\alpha|$, is the cardinality of its domain. Given two strings α and β , we write:

$$- \alpha \subseteq \beta \text{ if } |\alpha| \leq |\beta| \text{ and } \alpha(m) \leq \beta(m) \text{ for every } m < |\alpha|,$$

- $\alpha \sqsubseteq \beta$ if $|\alpha| \leq |\beta|$ and $\alpha(m) = \beta(m)$ for every $m < |\alpha|$,
- $\alpha \sqsubset \beta$ if $\alpha \sqsubseteq \beta$ and $\alpha \neq \beta$.

For every string β and every $m \leq |\beta|$, $\beta \upharpoonright m$ is the string $\alpha \sqsubseteq \beta$ with $|\alpha| = m$. If α is a string and $b \in \{0, 1\}$ then αb denotes the string of length $|\alpha| + 1$ such that $\alpha \sqsubset \alpha b$ and $\alpha b(|\alpha|) = b$. We fix an effective acceptable enumeration Φ_0, Φ_1, \dots of the Turing functionals. We fix also an effective acceptable enumeration $\varphi_0, \varphi_1, \dots$ of the Turing-computable unary functions. Finally, given two sets $A, B \subseteq \mathbb{N}$, A is *weak truth table* reducible to B , in short $A \leq_{wtt} B$, if there exists a number $e \in \mathbb{N}$ and a total computable function $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ such that:

- i) $\Phi_e^B = A$,
- ii) for every $x \in \mathbb{N}$, the computation of the e -th oracle Turing machine with oracle B on input x asks the oracle only numbers less than $\varphi(x)$.

In this case we say that (Φ_e, φ) *wtt-reduces* A to B . The weak truth-table reducibility is also known in literature as the *bounded Turing reducibility* \leq_{bT} .

3. Main result

Given any reducibility \leq_r and given any set $A \subseteq \mathbb{N}$, the r -degree of A is the class $\{B \subseteq \mathbb{N} : A \equiv_r B\}$, where $A \equiv_r B$ if and only if $A \leq_r B$ and $B \leq_r A$. A set A does not have subsets of higher r -degree if $A \not\leq_r B$ for every $B \subseteq A$. So a *wtt*-introimmune set does not have subsets of higher *wtt*-degree. In this section we prove the existence of a *wtt*-introimmune set in the class Δ_2^0 . Thus, for each known reducibility \leq_r strictly contained in \leq_T there are arithmetical r -introimmune sets. As for the arithmetical complexity we observe that for each reducibility \leq_r such that $\leq_1 \Rightarrow \leq_r$ there cannot be r -introimmune sets in Σ_1^0 , because such sets are immune. This follows from the fact that each 1-introimmune set is immune.

Proposition 3.1. *Each 1-introimmune set is immune.*

Proof. Let $A \subseteq \mathbb{N}$ be an infinite set and let us suppose that A is not immune. Then there exists an infinite recursive set $R \subseteq A$. Let $f : \mathbb{N} \rightarrow \mathbb{N}$

be a total one-one computable function such that $R = \{f(0), f(1), \dots\}$. Let us consider the infinite set

$$R_0 = \{f(\langle 0, n \rangle) : n \in \mathbb{N}\} \subseteq R.$$

Then,

$$A \setminus R_0 \subseteq A$$

and

$$|A \setminus (A \setminus R_0)| = |R_0| = \infty.$$

It follows that A is not 1-introimmune, because $A \leq_1 A \setminus R_0$ is witnessed by the total one-one computable function $g : \mathbb{N} \rightarrow \mathbb{N}$ defined in the following way:

1. $g(x) = x$ for every $x \notin R$, and
2. $g(f(\langle n, m \rangle)) = f(\langle n + 1, m \rangle)$ for every $n, m \in \mathbb{N}$.

It is routine to check that for every $x \in \mathbb{N}$, $x \in A \Leftrightarrow g(x) \in A \setminus R_0$. □

We know of the existence of m -introimmune sets in the class Π_1^0 [3, 4]. We leave as an open question the existence of *wtt*-introimmune sets in Π_1^0 .

Theorem 3.2. *There exists a wtt-introimmune set in Δ_2^0 .*

Proof. By the finite-extension method we construct a set A satisfying the following requirements for every $a, b, e \in \mathbb{N}$:

$$P_{2e} : |A| \geq e,$$

and

$N_{2\langle a, b \rangle + 1} : (\Phi_a, \varphi_b)$ does not *wtt*-reduce A to any $X \subseteq A$ with $|A \setminus X| = \infty$.

The satisfaction of all the requirements P_{2e} guarantees that A is infinite, while the satisfaction of all the requirements $N_{2\langle a, b \rangle + 1}$ guarantees that A is *wtt*-introimmune. □

3.1 Strategy

Set A will be constructed by infinitely many stages $s = 0, 1, \dots$. At every stage s we define the finite set A_s , and the final set will be

$$A = \lim_{s \rightarrow \infty} A_s,$$

with $A_s \subseteq A_{s+1}$ for every $s \geq 0$. Set A will be a subset of $\{h(n) : n \geq 0\}$, where $h : \mathbb{N} \rightarrow \mathbb{N}$ is a suitable *dominating* function.

Definition 3.3. (dominating function). A function $g : \mathbb{N} \rightarrow \mathbb{N}$ is *dominating* if for every total computable function $\varphi : \mathbb{N} \rightarrow \mathbb{N}$, $\varphi(n) < g(n)$ for almost every n .

Let $K = \{x \in \mathbb{N} : \varphi_x(x) \downarrow\}$ be the halting set, and let g be any increasing dominating K -computable function with $g(0) > 0$. Let us define the increasing sequence $(g^n(0) : n \geq 1)$ in the following way: $g^1(0) = g(0)$, and for every $n \geq 1$ $g^{n+1}(0) = g(g^n(0))$. Let us define for every $n \geq 1$

$$h(n) = g^n(0),$$

with $h(0) = 0$. Then, h is a dominating K -computable function which satisfies the following property.

Proposition 3.4. *Let $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ be any total computable function. Then for almost every $n \in \mathbb{N}$, for every $m \leq n$*

$$\varphi(h(m)) < h(n+1).$$

Proof. Given any such φ , let us consider the total computable function

$$\tilde{\varphi}(n) = \max\{\varphi(u) : u \leq n\}.$$

Let n_0 be such that for every $n \geq n_0$

$$\tilde{\varphi}(g^n(0)) < g(g^n(0)). \tag{1}$$

Then, for every $n \geq n_0$ and for every $m \leq n$

$$\varphi(h(m)) = \varphi(g^m(0)) \tag{2}$$

by definition of h , and

$$\varphi(g^m(0)) \leq \tilde{\varphi}(g^n(0)) \tag{3}$$

by $(g^n(0) : n \geq 1)$ increasing and by the definition of $\tilde{\varphi}$. Finally

$$\tilde{\varphi}(g^n(0)) < g(g^n(0)) = g^{n+1}(0) = h(n+1) \quad (4)$$

by (1) and by the definition of h . \square

3.2 Strategies to satisfy requirements

To satisfy each requirement P_{2e} we add an element to A at e opportune stages. The strategy to satisfy each requirement $N_{2\langle a,b \rangle+1}$ is essentially the method used in [1]. To satisfy $N_{2\langle a,b \rangle+1}$ means in particular to prevent (5):

$$(\exists X)[X \subseteq A \text{ and } |A \setminus X| = \infty \text{ and } \Phi_a^X = A]. \quad (5)$$

But (5) implies that there is an infinite sequence $(n_s : s \geq 0)$ of natural numbers such that

$$\Phi_a^X(h(n_s)) = A(h(n_s)) = 1 \text{ and } X(h(n_s)) = 0. \quad (6)$$

So, we wait for a stage $s+1$ at which

$$\varphi_b(h(s)) < h(s+1) \quad (7)$$

and for some $X \subseteq A_s \subseteq \{h(0), h(1), \dots, h(s-1)\}$ it is

$$\Phi_a^X(h(0)) = A_s(h(0)), \dots, \Phi_a^X(h(s-1)) = A_s(h(s-1)) \quad (8)$$

and

$$\Phi_a^X(h(s)) = 1. \quad (9)$$

Then, we force $\Phi_a^X(h(s))$ to be wrong by setting $A_{s+1}(h(s)) = 0$. Observe that by (7) and by $X \subseteq A_s \subseteq \{h(0), \dots, h(s-1)\}$ the computation of $\Phi_a^X(h(s))$ depends only on number less than or equal to $h(s)$.

3.3 Formalization

We formalize the above strategies and the construction of the set A . First, we define formally the conditions under which a requirement requires attention. Then, we will give an algorithm for the construction of the set A

by defining the actions needed to satisfy all the requirements. In order to better handle some proofs later we introduce first the following notation: given any string α , let X_α be the set

$$\{h(n) : n < |\alpha| \wedge \alpha(n) = 1\}.$$

From now on, Φ^α stands for Φ^{X_α} for each string α . The algorithm with which we will construct our set $A = \bigcup_{s \geq 0} A_s$ will generate by stages infinitely many strings $\alpha_0 \sqsubset \alpha_1 \sqsubset \dots$. The final set A will be

$$A = \lim_{s \rightarrow \infty} \alpha_s,$$

where α_s is the string obtained by the end of stage s with $|\alpha_s| = s$ and denoting $A_s = X_{\alpha_s}$.

3.3.1 Requirements requiring attention

Fix a stage $s + 1$, and let α_s be the string constructed by the end of stage s .

- Requirement P_{2e} requires attention at stage $s + 1$ if

$$|A_s| < e.$$

- Requirement $N_{2(a,b)+1}$ requires attention at stage $s + 1$ via the string α with $|\alpha| = |\alpha_s| = s$ if the following conditions hold.

C1: $\varphi_b(h(s)) < h(s + 1)$,

C2: $\Phi_a^\alpha(h(m))$ asks only elements less than $\varphi_b(h(m))$, for every $m < s$,

C3: $\alpha \subseteq \alpha_s$,

C4: (for every $m < s$), $[\Phi_a^\alpha(h(m)) = \alpha_s(m)]$,

C5: $\Phi_a^{\alpha^0}(h(s)) = 1$.

We describe the meaning of each condition. Condition **C1** makes the computation of $\Phi_a^\alpha(h(s))$ depending only on numbers less than or equal to $h(s)$. Condition **C2** says that (Φ_a, φ_b) could be a *wtt*-reduction. Condition **C3**

says that the set X_α is a subset of the constructed set A_s . Conditions **C4** and **C5** formalize (8) and (9), that is

$$\Phi_a^{X_\alpha}(h(0)) = A_s(h(0)), \dots, \Phi_a^{X_\alpha}(h(s-1)) = A_s(h(s-1))$$

and

$$\Phi_a^{X_\alpha}(h(s)) = 1.$$

3.3.2 Construction of the set A

We say that a N -requirement requires attention at stage $s+1$ if it requires attention at stage $s+1$ via some string α of length s . A requirement R_n has higher priority than a requirement R_m if $n < m$. At any stage $s+1$ a requirement R_n is *active* if it is the highest priority requirement requiring attention. The algorithm to construct the set A is the following.

Algorithm

- Stage 0. Set $\alpha_0 = \emptyset$.
- Stage $s+1$. Let α_s be the string constructed by the end of stage s , and let R_n be the active requirement. If n is even, then set $\alpha_{s+1} = \alpha_s 1$, otherwise set $\alpha_{s+1} = \alpha_s 0$.

End of algorithm

Set $A = \lim_{s \rightarrow \infty} \alpha_s$. The construction of A is by the finite extension method, thus for every stage $s \geq 0$ and for every $n < |\alpha_s|$, $\alpha_s(n) = A(h(n))$. Now we have to prove that the construction is correct, that is that each requirement is met and that $A \in \Delta_2^0$.

Lemma 3.5. *Every requirement requires attention at most finitely often and is met.*

Proof. By induction on the index n of the requirement R_n . Let $n \geq 0$ be given, and let s_0 be the minimum stage such that no requirement of higher priority than R_n requires attention after s_0 . Distinguish two cases on n .

- $R_n = P_{2e}$. Let us suppose that it requires attention at stage $s+1 > s_0$.

By hypothesis P_{2e} is active from stage $s + 1$ onwards. At each of these consecutive stages we add one element, so in at most $t \leq e$ stages starting from $s + 1$ the cardinality of A_{s+t} will be e , P_{2e} is satisfied and it will no longer require attention.

- $R_n = N_{2\langle a, b \rangle + 1}$. By Proposition 3.4 we can make the following further hypothesis on s_0 : for every $s \geq s_0$ and for every $m < s$,

$$\varphi_b(h(m)) < h(s). \quad (10)$$

From (10) we get the following

Claim 3.6. *For every string α and α' of length at least s_0 , if $\alpha \sqsubseteq \alpha'$, then*

$$(\forall m < |\alpha|)[\Phi_a^\alpha(h(m)) = \Phi_a^{\alpha'}(h(m))]. \quad (11)$$

Proof. Let $\alpha \sqsubseteq \alpha'$ with $|\alpha| \geq s_0$. For every $m < |\alpha|$ the computation of $\Phi_a^{X_\alpha}(h(m))$ can ask the oracle only numbers less than $\varphi_b(h(m)) < h(|\alpha|)$, where

$$X_\alpha \subseteq \{h(0), h(1), \dots, h(|\alpha| - 1)\}.$$

On the other hand, $\alpha \sqsubseteq \alpha'$ means that

$$\alpha = \alpha' \upharpoonright |\alpha|,$$

that is X_α is equal to $X_{\alpha'}$ up to $h(|\alpha| - 1)$. Therefore the two computations $\Phi_a^{X_\alpha}(h(m))$ and $\Phi_a^{X_{\alpha'}}(h(m))$ are equal for every $m < |\alpha|$. *End of proof of Claim 3.6.*

The proof that $N_{2\langle a, b \rangle + 1}$ requires attention at most finitely often is distributed in the following three claims¹.

Claim 3.7. *If $N_{2\langle a, b \rangle + 1}$ requires attention at stage $s + 1 > s_0$ via α , then for every s' with $s_0 \leq s' < s$ it holds that $\alpha(s') = A(h(s'))$.*

Proof. Let α_s be the string constructed by the end of stage s . For the sake of contradiction, let s' be the minimum such that $s_0 \leq s' < s$ and $\alpha(s') \neq A(h(s'))$. By hypothesis $N_{2\langle a, b \rangle + 1}$ requires attention via α at stage $s + 1$, thus by condition **C3**

¹ Technically, the proofs of these three claims are based on [2].

$$\alpha(s') \leq A(h(s')), \quad (12)$$

that is

$$\alpha(s') = 0 \text{ and } A(h(s')) = 1. \quad (13)$$

Let us consider $\beta = \alpha \upharpoonright s'$, that is

$$\beta \sqsubseteq \alpha \text{ and } |\beta| = s'. \quad (14)$$

We prove that $N_{2\langle a, b \rangle + 1}$ requires attention at stage $s' + 1$ via β , and this implies $\alpha_{s'+1} = \alpha_{s'}0$, that is $\alpha_{s'+1}(s') = 0$; but $\alpha_{s'+1} \sqsubseteq \alpha_s$, whence $\alpha_s(s') = 0$, that is $A(h(s')) = 0$, contradicting (13). In order to prove that $N_{2\langle a, b \rangle + 1}$ requires attention at stage $s' + 1$ via β it is enough to check that all the conditions **C1**, **C2**, **C3**, **C4** and **C5** hold for β and $\alpha_{s'}$ at stage $s' + 1$.

- **C1**: $\varphi_b(h(s')) < h(s' + 1)$ holds by (10) because $s' + 1 \geq s_0$.
- **C2**: $\Phi_a^\beta(h(m))$ asks only elements less than $\varphi_b(h(m))$ for every $m < s' < |\alpha|$, because **C2** holds at stage $s + 1$ w.r.t. α .
- **C3**: $\beta \sqsubseteq \alpha_{s'}$, because $\alpha_{s'} \sqsubseteq \alpha_s$, $\alpha \sqsubseteq \alpha_s$ and $\beta = \alpha \upharpoonright s'$.
- **C4**: $\beta \sqsubseteq \alpha$ with both the lengths of β and α at least s_0 , so by Claim 1 for every $m < |\beta|$

$$\Phi_a^\beta(h(m)) = \Phi_a^\alpha(h(m)). \quad (15)$$

Moreover, for every $m < |\beta|$

$$\Phi_a^\alpha(h(m)) = A(h(m)) \quad (16)$$

because **C4** holds at stage $s + 1$ w.r.t. α . Thus, by equations (15) and (16)

$$\Phi_a^\beta(h(m)) = A(h(m)) \quad (17)$$

for every $m < |\beta|$.

- **C5**: We observe first that $\beta 0 \sqsubseteq \alpha$, because by (13) it is $\alpha(s') = 0$ and by (14) it is $|\beta| = s'$. Then, by (10) the computation of $\Phi_a^{\beta 0}(h(s'))$ depends only on numbers $\leq h(s')$, which means that

$$\Phi_a^{\beta 0}(h(s')) = \Phi_a^\alpha(h(s')).$$

But by hypothesis $N_{2\langle a,b \rangle + 1}$ requires attention at stage $s + 1$, that is at stage $s + 1$ condition **C4** holds for every $m < s$, in particular for $m = s' < s$, so by the second equality of (13)

$$\Phi_a^\alpha(h(s')) = A(h(s')) = 1.$$

Therefore

$$\Phi_a^{\beta 0}(h(s')) = 1$$

and **C5** is satisfied. Hence, all the conditions **C1**, **C2**, **C3**, **C4** and **C5** are satisfied by β and $\alpha_{s'}$, so $N_{2\langle a,b \rangle + 1}$ requires attention at stage $s' + 1$ via β with $|\beta| = s'$. But as before observed this causes $A(h(s')) = 0$, contradicting (13). *End of proof of Claim 3.7.*

Claim 3.8. *Let us suppose that $N_{2\langle a,b \rangle + 1}$ requires attention via α at stage $s + 1 > s_0$, and let α' be such that $\alpha \sqsubset \alpha'$. Then, $N_{2\langle a,b \rangle + 1}$ does not require attention via α' .*

Proof. By hypothesis, at the end of stage $s + 1$ is

$$A(h(s)) = 0. \tag{18}$$

Let $s' > s$, and for the sake of contradiction let us suppose that $N_{2\langle a,b \rangle + 1}$ requires attention via α' at stage $s' + 1$. First, we note that it cannot be $\alpha 1 \sqsubseteq \alpha'$, because otherwise it would be

$$\alpha'(s) = 1$$

and by (18) $A(h(s)) = 0$, that is $\alpha_{s'}(s) = 0$, from which $\alpha' \not\sqsubseteq \alpha_{s'}$, contradicting condition **C3** $\alpha' \sqsubseteq \alpha_{s'}$ at stage $s' + 1$. Thus it has to be

$$\alpha 0 \sqsubseteq \alpha'. \tag{19}$$

Since by hypothesis $N_{2\langle a,b \rangle + 1}$ requires attention via α at stage $s + 1$ it follows that **C5** is satisfied, that is

$$\Phi_a^{\alpha 0}(h(s)) = 1.$$

On the other hand, by (19)

$$\Phi_a^{\alpha'}(h(s)) = \Phi_a^{\alpha 0}(h(s)) = 1.$$

But at stage $s' + 1$ $N_{2\langle a, b \rangle + 1}$ requires attention via α' , so by condition **C4** for $m = s < s'$

$$\Phi_a^{\alpha'}(h(s)) = A(h(s)),$$

that is $A(h(s)) = 1$, which contradicts (18). *End of proof of Claim 3.8.*

Claim 3.9. *For every string α of length s_0 , there is at most one string α' properly extending α such that $N_{2\langle a, b \rangle + 1}$ requires attention via α' .*

Proof. Let α be a string such that $|\alpha| = s_0$, and let α' and α'' be two strings properly extending α , that is

$$\alpha(m) = \alpha'(m) = \alpha''(m)$$

for every $m < s_0$. Let us suppose that $N_{2\langle a, b \rangle + 1}$ requires attention via α' at stage $s' + 1 > s_0$ and via α'' at stage $s'' + 1 > s_0$. Without loss of generality let us suppose that $|\alpha'| \leq |\alpha''|$. By Claim 3.7, for every t with $s_0 \leq t < s'$ it is

$$\alpha'(t) = A(h(t)) = \alpha''(t).$$

If $|\alpha'| = |\alpha''|$, then $\alpha' = \alpha''$. Otherwise $\alpha' \sqsubset \alpha''$, but this contradicts Claim 3.8. *End of proof of Claim 3.9*

Since there are 2^{s_0} string of length s_0 , by Claim 4 requirement $N_{2\langle a, b \rangle + 1}$ requires attention at most 2^{s_0} times after stage s_0 .

We prove now that $N_{2\langle a, b \rangle + 1}$ is met. For the sake of contradiction let us suppose that $N_{2\langle a, b \rangle + 1}$ is not met. This means that there exists $B \subseteq A$ such that

$$\Phi_a^B = A \tag{20}$$

and

$$|A \setminus B| = \infty. \tag{21}$$

Moreover, for every $x \in \mathbb{N}$ all the queries made in the computation $\Phi_a^B(x)$ are bounded by $\varphi_b(x)$. We proved that $N_{2\langle a, b \rangle + 1}$ requires attention at most finitely often. Hence, there is a minimum stage s_0 after which $N_{2\langle a, b \rangle + 1}$ does not require attention. By Proposition 3.4 and by (20) and (21) let $s + 1 > s_0$ such that the following three conditions are satisfied:

$$\varphi_b(h(s)) < h(s + 1), \tag{22}$$

$$\Phi_a^B(h(s)) = A(h(s)) = 1 \tag{23}$$

and

$$B(h(s)) = 0. \quad (24)$$

We show that $N_{2\langle a,b \rangle + 1}$ requires attention at $s + 1$, which is a contradiction. By (22) at stage $s + 1$ condition **C1** holds. Let us consider the string α of length s such that

$$\alpha(m) = B(h(m)) \quad (25)$$

for every $m < s$. String α satisfies all the conditions **C2**, **C3**, **C4** and **C5**:

- **C2**: $\Phi_a^\alpha(h(m))$ asks only elements less than $\varphi_b(h(m))$ for every $m < s$, because we are assuming that (Φ_a, φ_b) *wtt*-reduces A to B ;
- **C3**: $\alpha \subseteq \alpha_s$ because $B \subseteq A$;
- **C4**: by (20) and (25), for every $m < s$ $\Phi_a^\alpha(h(m)) = A(h(m)) = \alpha_s(m)$;
- **C5**: by (24) and (25), for every $m \leq s$

$$\alpha_0(m) = B(h(m)),$$

therefore by (23)

$$\Phi_a^{\alpha_0}(h(s)) = \Phi_a^B(h(s)) = 1.$$

Thus $N_{2\langle a,b \rangle + 1}$ requires attention at stage $s + 1$ via α , which is a contradiction. \square

It remains to prove that the set A is in Δ_2^0 .

Lemma 3.10. *A is in Δ_2^0 .*

Proof. We show that A is Turing reducible to the halting set K . It is enough to observe that oracle K suffices to find the active requirement at any stage, hence to generate the sequence $(\alpha_s : s \geq 0)$. We describe first an algorithm that at any stage $s + 1$ finds the active requirement and computes the extension α_{s+1} of α_s . Fix a stage $s + 1$ and let α_s be the string obtained by the end of stage s . Enumerate and check all the requirements R_0, R_1, \dots , stopping as soon as one of them satisfies the conditions under which it requires attention. For the part concerning the check, let R_n be a requirement of the above list and distinguish two cases:

- $R_n = P_{2n}$. It is decidable whether or no P_{2e} requires attention, and in this case oracle K is unnecessary.
- $R_n = N_{2\langle a,b \rangle + 1}$. With oracle K compute first $h(s)$ and $h(s+1)$. Let $F(a, b, X_\alpha, \alpha_s, s, h(s), h(s+1))$ be the formula obtained by the conjunction of the formulas expressing conditions **C1**, **C2**, **C3**, **C4** and **C5** with X_α in place of α . Then, $N_{2\langle a,b \rangle + 1}$ requires attention at stage $s+1$ if the formula

$$(\exists \alpha)[|\alpha| = |\alpha_s| \wedge F(a, b, X_\alpha, \alpha_s, s, h(s), h(s+1))] \quad (26)$$

is true. In (26) the existential quantifier on the oracle variable α is bounded, and for each such α oracle K suffices to compute the relative finite set X_α . All the values $h(m)$ for $m < s$ required in the formula are also computable with K . Finally, observe that $F(a, b, X_\alpha, \alpha_s, s, h(s), h(s+1))$ is a Σ_1^0 formula, so oracle K is enough to test its truth. This shows that K suffices to generate $(\alpha_s : s \geq 0)$. To decide A , given any $x \in \mathbb{N}$ generate the sequence $\alpha_0, \alpha_1, \dots, \alpha_{m+1}$, where m is the minimum such that $h(m) \geq x$. If $h(m) > x$ then reject x . Otherwise, accept x if and only if $\alpha_{m+1}(m) = 1$.

This concludes the proof of Lemma 3.10 and the proof of the theorem. \square

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