REPORTS ON MATHEMATICAL LOGIC
53 (2018), 111–142
doi: 10.4467/20842589RM.18.007.8840

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THE *-PRIKRY CONDITION

A b s t r a c t. In this paper we isolate a property for forcing notions, the *-Prikry condition, that is similar to the Prikry condition but that is topological: A forcing $\mathbb{P}$ satisfies it iff for every $p \in \mathbb{P}$ and for every open dense $D \subseteq \mathbb{P}$, there are $n \in \omega$ and $q \leq^* p$ such that for any $r \leq q$ with $l(r) = l(q) + n$, $r \in D$, for some length notion l. This is implicit in many proofs in literature. We prove this for the tree Prikry forcing and the long extender Prikry forcing.

The key notion in this paper is Prikry forcing and its generalizations. Prikry forcing adds a cofinal ω sequence to a measurable cardinal, and its generalizations usually add more of such ω-sequences, in different settings. What all these forcings have in common is the "Prikry condition", a property that is fundamental in proving the typical characteristics of Prikry forcings (for example the fact that they do not add bounded sets). We call "*-Prikry condition" a slight modification of the original condition: it is needed many times in literature (see for example [13] and [8], yet

Received 16 October 2017. Revised 2 June 2018
Keywords and phrases: Prikry forcing.
AMS subject classifications: 03E55, 03E05, 03E35(03E45).

it is never defined. For example, it is used in [4], [14] and [15] to prove λ-goodness of Prikry-type forcing, essential in the study of forcing with I0,[1] in [8] as a means to prove the original Prikry condition, and in [13] to prove that "simple" equivalence relation naturally extend to generic extensions. Moreover, while the original Prikry condition is a concept related to forcing, the *-Prikry condition is a topological property, that depends only on the topology of the partial orders involved.

The proof of the *-Prikry condition for the original Prikry forcing is classic, but the more sophisticated the forcing is, the more difficult it is to find a proof of the *-Prikry condition for it (see for example the long section 5 in [14] and [15]). Sometimes it is proved from scratch for a particular forcing (see [9]), other times it is just assumed (see [13]). In this paper we want to outline a strategy to prove it that can be applied to many (arguably all the "classic" ones) Prikry-like forcings, and we are going to apply this strategy to the tree Prikry forcing and to the Gitik-Magidor long extender Prikry forcing. We feel that these two examples already cover much ground: for example supercompact Prikry forcings are just measure forcings, so the techniques from the classical and the tree Prikry forcing cases apply, and with almost no further effort, the strategy works also for diagonal supercompact Prikry forcing, both the original by Gitik-Sharon [7] and the variation by Neeman [10]. On the other hand Gitik-Magidor long extender Prikry forcing exemplifies all the other Prikry forcings that add many Prikry sequences via an extender.

The author would like to thank the FWF (Austrian Science Fund) for its generous support through project M 1514-N25, the MIUR (Ministry of Instruction, University and Research) for its support through the program "Rita Levi Montalcini 2013" and the kind hospitality of the Kurt Gödel Research Center, Beijing Normal University and the Chinese Academy of Sciences.

1. Prikry and tree Prikry forcing

Let κ be a measurable cardinal, with normal measure U.

[1] For more on I0 and beyond, see for example [1], [2], or [3].

Definition 1.1. The Prikry forcing $\mathbb{P}$ on κ via U is the set of conditions (s, A) such that $s \in \kappa^{<\omega}$, $A \in U$ and $\max s < \min A$. We say that $(t, B) \leq (s, A)$ if $s \subseteq t$, $B \subseteq A$ and $t \setminus s \subseteq A$.

We introduce some basic notation and terminology about trees. Trees are a typical structure that is investigated in combinatorics. Let α be an ordinal. For any $s \in [\alpha]^n$, $\mathrm{lh}(s) = n$. A (descriptive set-theoretical) tree on α is a subset of $[\alpha]^{<\omega}$ closed under initial segments. If T is a tree, for any $s \in T$, denote $T_s = \{t \in T : t \subseteq s \vee s \subseteq t\}$, $\mathrm{Suc}_T(s) = \{\beta \in \alpha : t^\frown\langle\beta\rangle \in T\}$, $T_{s^\frown} = \{t : s^\frown t \in T\}$ and finally for any $n \in \omega$, $\mathrm{Lev}_n(T) = \{s \in T : \mathrm{lh}(s) = n\}$. The trunk of T is the longest s such that $\forall t \in T\ s \subseteq t$.

Prikry forcing is useful because it is a very "delicate" forcing [5]: it does not add bounded subsets of κ, and is κ^+-cc, so it does not change the cardinal structure above κ. In other words, it makes κ singular while changing the universe as little as possible.

We now define the tree Prikry forcing (see [5] Section 1.2):

Definition 1.2. Let κ be a measurable cardinal. Fix U a measure on κ. The tree Prikry forcing $\mathbb{P}$ is the set of conditions $p = (s_p, T^p)$, where s_p is a finite sequence of ordinals in κ, and T^p is a (descriptive set-theoretical) tree of finite increasing sequences in κ with stem s_p, such that for any $t \in T^p$, $\mathrm{Suc}_{T^p}(t) \in U$. We say that $p \leq q$ if $s_p \supseteq s_q$ and $T^p \subseteq T^q$. We say that $p \leq^* q$ if $p \leq q$ and $s_p = s_q$. For any $p \in \mathbb{P}$ and $t \in T^p$, we write $p \oplus t$ for $(t, (T^p)_t)$.

The difference between the two forcings is minimal: the only difference is that the standard Prikry forcing uses a normal ultrafilter, while for tree Prikry forcing normality is not needed. In most of the applications the ultrafilters are normal, and in these cases the two forcing notions produce the same results (compare [5] Theorems 1.10 and 1.25), so using one or the other is often a matter of taste.

At first glance, Prikry forcing does not seem delicate at all, as it is not even ω-closed. But $\leq^*$ is actually κ-closed, and the crucial notion that makes everything work is the Prikry condition:

Lemma 1.3 (Prikry condition)**.** *Let $\mathbb{P}$ be a Prikry forcing or a tree Prikry forcing on κ, and let σ be a statement of the forcing language. Then for any $p \in \mathbb{P}$ there exists a $q \leq^* p$ such that $q \Vdash \sigma$ or $q \Vdash \neg\sigma$.*

In many cases (obviously in [4], but also in [14], for example), the Prikry condition is not that useful, but a slight variation actually it is.

Definition 1.4. Let $(\mathbb{P}, \leq)$ be a forcing notion, and $\leq^* \subseteq \leq$ be an order. We say that $\mathbb{P}$ satisfies the *-Prikry condition if:

- there exists a function $l : \mathbb{P} \to \omega$ measuring the length of the condition, that is $l(1_{\mathbb{P}}) = 0$ and for any $p, q \in \mathbb{P}$, if $p \leq q$ then $l(p) \geq l(q)$, and $p \leq^* q$ iff $l(p) = l(q)$;
- for every $p \in \mathbb{P}$ and for every open dense $D \subseteq \mathbb{P}$, there are $n \in \omega$ and $q \leq^* p$ such that for any $r \leq q$ with $l(r) = l(q) + n$, we have that $r \in D$.

Such condition is usually satisfied by forcings that satisfy the Prikry condition, and the proofs tend to be very similar. In the Prikry case the combinatorial core of it is the Rowbottom Theorem:

Theorem 1.5 (Rowbottom, [11]). *Suppose that κ is measurable and U is a normal ultrafilter on κ. Then for any $\gamma < \kappa$ and any $f : [\kappa]^{<\omega} \to \gamma$, there is a set in U homogeneous for f.*

Lemma 1.6. *Prikry forcing on κ has the *-Prikry condition.*

Proof. It is actually a very well known fact, see for example Lemma 1.13 in [5]. For completeness, we write the proof here.

Let U be the ultrafilter that generates the Prikry forcing $\mathbb{P}$. For any $p = (s, A) \in \mathbb{P}$, define $l(p) = \mathrm{lh}(s)$. Let $p = (s, A)$ and $D \subseteq \mathbb{P}$ be open dense. Let $h : [A]^{<\omega} \to 2$ be the partition such that $h(t) = 1$ iff there exists a C such that $(s^\frown t, C) \in D$. By the Rowbottom Theorem, there exists $B \subseteq A$ in U homogeneous for h, i.e. such that for every $n \in \omega$ and $s_1, s_2 \in [B]^n$, $h(s_1) = h(s_2)$. Since D is dense, we get that there exists $n \in \omega$ such that for any $m > \omega$ and for all $t \in [B]^m$ we have $h(t) = 1$. Let for each $t \in [B]^{<\omega}$, $C_t \in U$ be such that $(s^\frown t, C_t) \in D$, and C be the diagonal intersection of all the C_t. Then $(s, B \cap C)$ and n witness that the *-Prikry condition holds for (s, A). $\square$

The following is the most basic non-immediate example, and the method used in the proof is also the base for the more sophisticated methods in the next sections:

Lemma 1.7. *Let κ be a measurable cardinal. Then the tree Prikry forcing $\mathbb{P}$ on κ has the *-Prikry condition.*

Proof. Fix U a measure on κ. For any $p = (s_p, T^p) \in \mathbb{P}$, we define $l(p) = \mathrm{lh}(s_p)$. The proof is in three steps:

- in the first step, we modify the tree T^p so that if some condition $r \leq p$ is in D, then all the conditions $t \leq p$ such that $s_r = s_t$ are in D;
- in the second step, we modify the previous tree, so that if some condition $r \leq p$ with $T^r = (T^p)_{s_r}$ is in D, then all the conditions $t \leq p$ with $T^t = (T^p)_{s_t}$ and $l(r) = l(t)$ are in D;
- in the third step, we put the two claims together, and prove the lemma.

Claim 1.8 (First step)**.** *For any D open dense set and for any $p \in \mathbb{P}$, there exists $q \leq^* p$ such that if there exists $r = (s_r, T^r) \leq q$ (i.e. $r \leq^* q \oplus s_r$) such that $r \in D$, then $q \oplus s_r \in D$.*

Proof of claim. For any $p = (s_p, T^p) \in \mathbb{P}$, then T^p is isomorphic to the complete tree TC, and $1_{\mathbb{P}} = (\langle\rangle, TC)$, therefore $\mathbb{P}_p = \{q \in \mathbb{P} : q \leq p\}$ is isomorphic to $\mathbb{P}$, so we can suppose $p = 1_{\mathbb{P}}$. The proof is by induction. Informally, we consider $T^{1_{\mathbb{P}}}$, and we restrict it by asking, at each level, whether there is a possible way to shrink it to reach D: if there is, then we just shrink it; otherwise we do nothing. More formally, we define S_n, trees on κ, for $n \in \omega$ by induction:

- if there exists a $r \leq^* 1_{\mathbb{P}}$ such that $r \in D$, then let S_0 be T^r; otherwise $S_0 = T^{1_{\mathbb{P}}}$ (note that in the first case $(\langle\rangle, S_0) = r \in D$);
- given S_n, first we require that $S_{n+1} \restriction [\kappa]^{n+1} = S_n \restriction [\kappa]^{n+1}$; then for any $s \in \mathrm{Lev}_{n+1}(S_n)$, if there exists a $r \leq (\langle\rangle, S_n)$ such that $s_r = s$ and $r \in D$, then let $(S_{n+1})_s = (T^r)_s$; otherwise $(S_{n+1})_s = (S_n)_s$ (note that in the first case $(s_r, (S_{n+1})_{s_r}) = r \in D$).

Let $S = \bigcap_{n\in\omega} S_n$ and $q = (\langle\rangle, S)$. Then q is as desired: since the n-th level is changed just in the first n steps, we have that for any $s \in S$ of length $n \in \omega$, $\mathrm{Suc}_S(s) = \bigcap_{m \leq n} \mathrm{Suc}_{S_m}(s) \in U$, therefore $q \in \mathbb{P}$. Let $t \leq^* q \oplus s_t$, i.e. $s_t \in S$ and $T^t \subseteq S$, with $t \in D$. Suppose $\mathrm{lh}(s_t) = n$. Then also $s_t \in S_n$, as $S \restriction [\kappa]^{n+1} = S_n \restriction [\kappa]^{n+1}$, so $t \leq (\langle\rangle, S_n)$. This means that in the construction the first case was true, therefore $q \oplus s_t = (s_t, S_{s_t}) \leq (s_t, (S_{n+1})_{s_t}) \in D$. $\square$

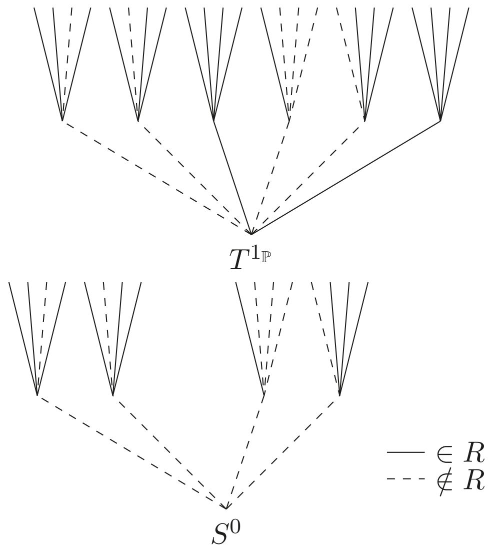


Figure 1: We draw with a dashed line the nodes not in R, and with a normal line the nodes in R. In this example, the "majority" of nodes of length 1 of $T^{1_{\mathbb{P}}}$ are not in R, therefore $A^0_{\langle\rangle} = \mathrm{Suc}_{T^{1_{\mathbb{P}}}}(\langle\rangle) \setminus B^0_{\langle\rangle} \in U$ and we delete all the nodes that are not in $A^0_{\langle\rangle}$ and everything above to construct S^0.

Claim 1.9 (Second step)**.** *For any D open dense set and for any $p \in \mathbb{P}$, there exist $q \leq^* p$ and $n \in \omega$ such that for any $s_1, s_2 \in T^q$ such that $l(s_1) = l(s_2) = n$, $q \oplus s_1 \in D$ iff $q \oplus s_2 \in D$.*

Proof. We can still assume $p = 1_{\mathbb{P}}$. Let $R = \{s \in T^{1_{\mathbb{P}}} : p \oplus s \in D\}$.

Informally, we are climbing up level by level, deleting at each level either the branches that are in R or the ones that are not, so that the sets of successors are still in U. We define by induction the trees S^n and $S^{n,m}$, with $n, m \in \omega$. The first step, then, will be simple: let $B^0_{\langle\rangle} = \{\delta \in \mathrm{Suc}_{T^{1_{\mathbb{P}}}}(\langle\rangle) : \langle\delta\rangle \in R\}$. Then exactly one among $B^0_{\langle\rangle}$ and $\mathrm{Suc}_{T^{1_{\mathbb{P}}}}(\langle\rangle) \setminus B^0_{\langle\rangle}$ is in U, and we call it $A^0_{\langle\rangle}$. Then let $\langle\mu_0, \ldots, \mu_l\rangle \in S^0$ iff $\mu_0 \in A^0_{\langle\rangle}$ (see Figure 1).

Note that in S^0, $\mathrm{Suc}_{S^0}(\langle\rangle) = A^0_{\langle\rangle} \subseteq \mathrm{Suc}_{T^{1_{\mathbb{P}}}}(\langle\rangle)$, while for all $s \in S^0$ of length ≥ 1, $\mathrm{Suc}_{S^0}(s) = \mathrm{Suc}_{T^{1_{\mathbb{P}}}}(s)$, and the sequences in S^0 of length 1 are either all in R or all outside.

The second step adds another layer of complexity. First, for any $\langle\mu\rangle \in S^0$ we restrict its successors so that they are either all in R or all outside R. Therefore let

$$B^1_{\langle\mu\rangle} = \{\delta \in \mathrm{Suc}_{S^0}(\langle\mu\rangle) : \langle\mu, \delta\rangle \in R\}$$

for any $\langle\mu\rangle \in \mathrm{Suc}_{S^0}(\langle\rangle)$. Then exactly one among $B^1_{\langle\mu\rangle}$ and $\mathrm{Suc}_{S^0}(\langle\mu\rangle) \setminus B^1_{\langle\mu\rangle}$ is in U, and we call it $A^1_{\langle\mu\rangle}$. Now define $S^{1,0}$ so that $\langle\mu_0, \dots, \mu_l\rangle \in S^{1,0}$ iff $\mu_0 \in \mathrm{Suc}_{S^0}(\langle\rangle)$ and $\mu_1 \in A^1_{\langle\mu_0\rangle}$ (see Figure 2). Note that for all $s \in S^{1,0}$ with $\mathrm{lh}(s) = 1$, $\mathrm{Suc}_{S^1_0}(s) = A^1_{\langle s(0)\rangle} \subseteq \mathrm{Suc}_{S^0}(s)$, while if $\mathrm{lh}(s) \neq 1$ then $\mathrm{Suc}_{S^1_0}(s) = \mathrm{Suc}_{S^0}(t)$.

This is not enough. One by one, each of the 2-sequences that share the same root are either all in R or all outside, but it can be that all the 2-sequences that start with μ_1 are in R, and all the 2-sequences that start with μ_2 are not in R. Therefore we must choose only the μ's that give a consistent result.

Let

$$B^1_{\langle\rangle} = \{\mu \in \mathrm{Suc}_{S^0}(\langle\rangle) : \mathrm{Suc}_{S^{1,0}}(\langle\mu\rangle) = B^1_{\langle\mu\rangle}\},$$

i.e. the set of μ's such that for any $\delta \in \mathrm{Suc}_{S^{1,0}}(\langle\mu\rangle)$, $\langle\mu, \delta\rangle \in R$. Then exactly one among $B^1_{\langle\rangle}$ and $\mathrm{Suc}_{S^0}(\langle\rangle) \setminus B^1_{\langle\rangle}$ is in U. Let $A^1_{\langle\rangle}$ be it. Now define $S^{1,1} = S^1$ as $\langle\mu_0, \dots, \mu_l\rangle \in S^1$ iff $\langle\mu_0, \dots, \mu_l\rangle \in S^{1,0}$ and $\mu_0 \in A^1_{\langle\rangle}$ (see Figure 2). Note that for all $s \in S^1$, if $\mathrm{lh}(s) = 0$ then $\mathrm{Suc}_{S^1}(s) = A^1_{\langle\rangle} \subseteq \mathrm{Suc}_{S^{1,0}}(s)$, otherwise $\mathrm{Suc}_{S^1}(s) = \mathrm{Suc}_{S^{1,0}}(s)$. The sequences in S^1 of length 2 are either all in R or all outside it.

By induction the construction continues level-by-level, each time starting with $S^{n+1,0} \subseteq S^n$, and then going down to S^{n+1}, a tree such that all the $n+1$-branches are either all in R or all outside it. More formally, suppose S^n is defined. For all $t \in S^n$, $\mathrm{lh}(t) = n+1$, define $B^{n+1}_t = \{\delta \in \mathrm{Suc}_{S^n}(t) : t^\frown\langle\delta\rangle \in R\}$. Then exactly one among B^{n+1}_t and $\mathrm{Suc}_{S^n}(t) \setminus B^{n+1}_t$ is in U, and we call it A^{n+1}_t. Define $\langle\mu_0, \dots, \mu_l\rangle \in S^{n+1,0}$ iff $\langle\mu_0, \dots, \mu_l\rangle \in S^n$ and $\mu_{n+1} \in A^{n+1}_{\langle\mu_0,\dots,\mu_n\rangle}$. Note that for all $s \in S^{n+1,0}$, $\mathrm{lh}(s) = n+1$,

$$\mathrm{Suc}_{S^{n+1,0}}(s) = A^{n+1}_s,$$

otherwise $\mathrm{Suc}_{S^{n+1,0}}(s) = \mathrm{Suc}_{S^n}(s)$.

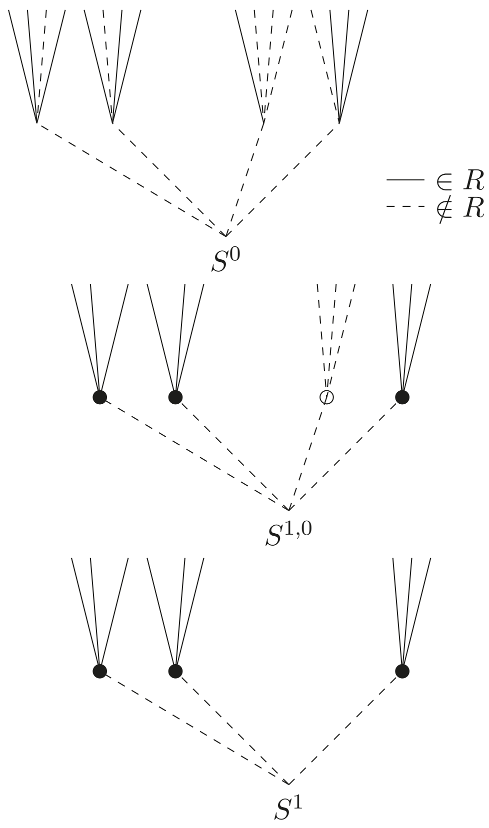


Figure 2: We continue the example of the previous figure. The first passage is, for each node of length one of S^0, to prune the successor nodes that are in the "minority", so that in $S^{1,0}$ any node of length 1 has either all the immediate successors in R, or all the immediate successors not in R. We tag with a black point the nodes in the first case, and with a white point the nodes in the second case. The nodes tagged with a black point are those in $B^1_{\langle\rangle}$. Now, the "majority" of nodes have the black point, so $A^1_{\langle\rangle} = B^1_{\langle\rangle} \in U$.

Let $t \in \mathrm{Lev}_m S^n$ and suppose that $S^{n+1,n-m}$, B_s^{n+1} and A_s^{n+1} are defined for all $s \in S^{n+1}$ with $\mathrm{lh}(s) = m+1$. Let

$$B_t^{n+1} = \{\delta \in \mathrm{Suc}_{S^{n+1}}(t) : \mathrm{Suc}_{S^{n+1,n-m}}(t^\frown\langle\delta\rangle) = B_{t^\frown\langle\delta\rangle}^{n+1}\}.$$

Then exactly one among B_t^{n+1} and $\mathrm{Suc}_{S^{n+1}}(t) \setminus B_t^{n+1}$ is in U. Let A_t^{n+1} be it.

Suppose A_t^{n+1} is defined for all $t \in S^{n+1}$ of length m. Then $\langle \mu_0, \dots, \mu_l \rangle \in S^{n+1,n+1-m}$ iff $\langle \mu_0, \dots, \mu_l \rangle \in S^{n+1,n-m}$ and $\mu_i \in A^{n+1}_{\langle \mu_0, \dots, \mu_m \rangle}$. Note that for all $s \in S^{n+1,n+1-m}$ of length $n+1-m$,

$$\mathrm{Suc}_{S^{n+1,n+1-m}}(s) = A_s^{n+1},$$

otherwise $\mathrm{Suc}_{S^{n+1,n+1-m}}(s) = \mathrm{Suc}_{S^{n+1,n-m}}(s)$. Call $S^{n+1,n+1} = S^{n+1}$. Then all the sequences in S^{n+1} of length $n+1$ either are all in R or all outside it.

Now, let $S = \bigcap_{n\in\omega} S^n$. The last remark is sufficient to prove the claim. We prove that $(\langle\rangle, S) \in \mathbb{P}$. It suffices to prove that for any $t \in S$, $\mathrm{Suc}_S(t) \in U$. So let $t \in S$, $\mathrm{lh}(t) = n$. Then $\mathrm{Suc}_S(t)$ will be modified in the construction of S only in the stages $S^{n+i,n}$, with $i \in \omega$, and $S^{n,0}$, therefore

$$\mathrm{Suc}_S(t) = \bigcap_{i\in\omega} A_t^{n+i},$$

that is a countable intersection of elements of U, and therefore in U. $\square$

Claim 1.10 (Third step)**.** *For any $p \in \mathbb{P}$ and for any D open dense set there exist a $q \leq^* p$ and an $n \in \omega$ such that for any $t \leq q$ with* $\mathrm{lh}(t) = n$*, $t \in D$.*

Proof of claim. Pick a $q' \leq^* p$ as the first claim and a $q \leq^* q'$ as the second claim. By density, there exists a $r \leq q$, $r \in D$. Let $n = \mathrm{lh}(r)$. Then by the first claim $q' \oplus s_r \in D$. Since $r \leq q$, $s_r \in T^q$, and since $q \leq q'$, $q \oplus s_r = (s_r, (T^q)_{s_r}) \leq q' \oplus s_r \in D$. By the second claim, then, all the extensions of q of the same length of r are in D. $\square$

2. Extender-based Prikry forcing

The three-steps approach can be used in many other situations than the tree Prikry forcing. As an example, we show how it is useful for the Gitik-

Magidor long extender Prikry forcing. As the forcing is decidedly more complex, the proof is more involved, but the main steps are the same:

- In every Prikry forcing the conditions have componenents of two types: a base, made of a sequence or a set of sequences (that is an approximation of the generic we want) and a structure based on measures (that describes the possible extensions of the base). Let p be a condition.
- We extend p to a q' that is somewhat universal for having an extension in D, in the sense that if an extension of q' is in D, then the weakest extension of q' of the same length is in D; this is done by shrinking the "measure" part so to be compatible with all the possible extensions in D.
- We extend q' to a q so that, for each level, either all the conditions that extend q are in D, or they are not in D.
- Finally there should be an extension of q in D, and this should give enough information for the proof of the *-Prikry condition.

Gitik-Magidor long extender Prikry forcing was introduced by Gitik and Magidor, and the reader can find an exhaustive description in [5], Section 3. The aim of the forcing is to add many Prikry sequences to a strong enough cardinal λ, blowing up its power while not changing the power function below it. So if the ground model satisfies GCH, then in the forcing extension λ is singular and it is the first cardinal on which GCH fails. This is more difficult than just having λ singular and $2^\lambda > \lambda^+$: the proof for this second statement usually consists in taking λ measurable, forcing $2^\lambda > \lambda^+$ and then adding a Prikry sequence to λ. But Dana Scott [12] proved that if λ is measurable and $2^\lambda > \lambda^+$, then for a measure one set below λ, $2^\kappa > \kappa^+$, therefore this method would not give the first failure of GCH on λ. The solution is to exploit the extender structure of the cardinal to add many Prikry sequences, at the same time blowing up the power and changing the cofinality.

The main limitation of this forcing is that it needs a long extender, and more precisely an extender with length η bigger than λ^+, because it is a forcing that adds exactly η Prikry sequences to λ. The same forcing will still work with short extenders, but it will not give any new result.

Gitik defined a short extender version that has the same effect than the long extender version (see [6]), but it is a hybrid Cohen-Prikry forcing, so we do not feel it belongs to this initial analysis of *-Prikry condition for Prikry forcings. It could be a future further development to check whether also this hybrid forcing satisfies the *-Prikry condition or a generalization of it.

Definition 2.1. Let κ and γ be cardinals. Then κ is γ-strong iff there is a $j : V \prec M$ such that $\mathrm{crt}(j) = \kappa$, $\gamma < j(\kappa)$ and $V_\gamma \subseteq M$.

The definition below follows closely the treatment in [5], Section 3, and we refer the reader to such paper for any undefined notion.

Suppose GCH, and let λ be a $\lambda + 2$-strong cardinal. By the theory of the extenders, this means that there is an extender on λ of length λ^{++}, that is a system of ultrafilters defined in this way: for any $\alpha < \lambda^{++}$, define a λ-complete normal ultrafilter on λ as $X \in U_\alpha$ iff $\alpha \in j(X)$. Note that for $\alpha < \lambda$ this is trivial, and for $\alpha = \lambda$ this is the usual ultrafilter for measurability.

These ultrafilters form a structure. Since we would like this paper to be approachable even to readers not familiar with extenders, we are going to describe this structure without proofs, as a black box. All the proofs are in [5], Section 3. The forcing notion will be built on this structure.

For any $\alpha, \beta < \lambda^{++}$, define $\alpha \leq_E \beta$ iff $\alpha \leq \beta$ and for some $f \in {}^\lambda\lambda$, $j(f)(\beta) = \alpha$. Then $\langle \lambda^{++}, \leq_E \rangle$ has the following properties:

- It is a λ^{++}-directed order.
- $\lambda \leq_E \alpha$ for any $\alpha < \lambda^{++}$.
- We can define "projections" from different U_α's: for any $\beta \leq_E \alpha < \lambda^{++}$ there are $\pi_{\alpha\beta} : \lambda \to \lambda$ such that if $\gamma < \beta \leq \alpha < \lambda^{++}$, and $\gamma, \beta \leq_E \alpha$, then $\{\nu < \lambda : \pi_{\alpha\beta}(\nu) > \pi_{\alpha\gamma}(\nu)\} \in U_\alpha$.
- For any $\gamma \leq_E \beta \leq_E \alpha$ there is an $A \in U_\alpha$ so that for avery $\nu \in A$, $\pi_{\alpha\gamma}(\nu) = \pi_{\beta\gamma}(\pi_{\alpha\beta}(\nu))$.
- (Full commutativity at λ) For every $\alpha, \beta < \lambda^{++}$ and $\mu < \lambda$, if $\alpha \geq_E \beta$ then $\pi_{\alpha\lambda}(\mu) = \pi_{\beta\lambda}(\pi_{\alpha\beta}(\mu))$.

- (Independence of the choice of projection to λ): for every $\alpha, \beta < \lambda^{++}$ and $\mu < \lambda$, $\pi_{\alpha\lambda}(\mu) = \pi_{\beta\lambda}(\mu)$[2].

For more readability, we write $\pi_{\alpha\lambda}$ just $\pi_{\alpha,0}$.

Because of the independence of the choice of projection to λ, for any $\alpha, \beta < \lambda^{++}$ and any $\nu < \lambda$, $\pi_{\alpha,0}(\nu) = \pi_{\beta,0}(\nu)$. For any $\nu < \lambda$ and $\lambda < \alpha < \lambda^{++}$, then, let us denote $\pi_{\alpha,0}(\nu)$ by ν^0. By a $^\circ$-increasing sequence of ordinals we mean a sequence $\langle \nu_0, \dots, \nu_n \rangle$ of ordinals below λ such that $\nu_0^0 < \dots < \nu_n^0$. We say that μ is permitted for $\langle \nu_0, \dots, \nu_n \rangle$ iff $\langle \nu_0, \dots, \nu_n, \mu \rangle$ is $^\circ$-increasing, i.e., $\mu^0 > \nu_i^0$ for all $i = 0 \dots n$. By the full commutativity at λ, for any $\beta \leq_E \alpha$, μ is permitted for a sequence iff $\pi_{\alpha\beta}(\mu)$ is permitted for the same sequence.

With some cosmetic change to the U_α's, we can suppose that if $A \in U_\alpha$, $\mu_0, \mu_1 \in A$ and $\mu_0^0 < \mu_1^0$, then $|\{\mu \in A : \mu^0 = \mu_0^0\}| < \mu_1^0$ (see [5] the third and fourth paragraph before Definition 3.6).

Definition 2.2. The forcing $\mathbb{P}$ consists of all p of the form

$$\{\langle \gamma, p^\gamma \rangle : \gamma \in g \setminus \{\max(g)\}\} \cup \{\langle \max(g), p^{\max(g)}, T \rangle\},$$

where

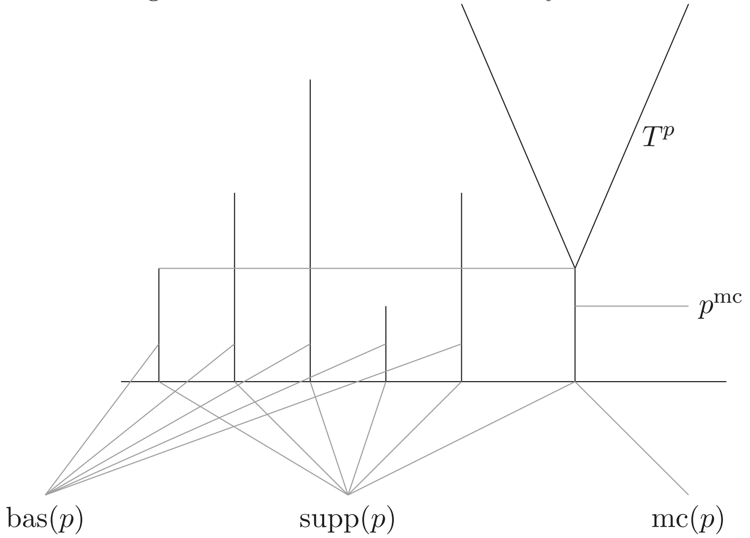
1. $g \subseteq \lambda^{++}$ is of cardinality $\leq \lambda$, has a maximal element according to $\leq_E$ and $0 \in g$.
2. for $\gamma \in g$, p^γ is a finite $^\circ$-increasing sequence of ordinals $< \lambda$.
3. T is a tree, with trunk $p^{\max(g)}$, consisting of $^\circ$-increasing sequences. All the splittings in T are required to be on sets in $U_{\max(g)}$, i.e., for every $\eta \in T$, if $\eta \geq p^{\max(g)}$ then the set
$$\mathrm{Suc}_T(\eta) = \{\mu < \lambda : \eta^\frown \langle \mu \rangle \in T\} \in U_{\max(g)}.$$
Also require that for $\eta_1 \geq_T \eta_2 \geq_T p^{\max(g)}$, $\mathrm{Suc}_T(\eta_1) \subseteq \mathrm{Suc}_T(\eta_2)$.
4. For every $\mu \in \mathrm{Suc}_T(p^{\max(g)})$, $|\{\gamma \in g : \mu \text{ is permitted for } p^\gamma\}| \leq \mu^0$.
5. For every $\gamma \in g$, $\pi_{\max(g),\gamma}(\max(p^{\max(g)}))$ is not permitted for p^γ.

[2] There are also other properties satisfied by the structure $\langle \lambda^{++}, \langle U_\alpha : \alpha < \lambda^{++} \rangle, \leq_E \rangle$. Gitik in [5] calls this structure a nice system. To prove the *-Prikry condition, such details are not needed, but they are essential to prove the right consequences of the forcing.

6. $\pi_{\max(g),0}$ projects $p^{\max(g)}$ onto p^0 (so $p^{\max(g)}$ and p^0 are of the same length).

Let us denote g by $\mathrm{supp}(p)$, $\max(g)$ by $\mathrm{mc}(p)$, T by T^p, $p^{\max(g)}$ by p^{mc} and $\mathrm{bas}(p) = p \restriction (\mathrm{supp}(p) \setminus \mathrm{mc}(p))$.

We can imagine a condition of $\mathbb{P}$ in this way:



Clearly, the picture ignores many elements (especially the π's), but it is a good approximation. The idea is that for any $\gamma < \lambda^{++}$, we are building a generic for the tree Pikry forcing $\mathbb{P}_\gamma$ via U_γ. We can think of $(p^\gamma, \pi''_{mc(p),\gamma}T^p)$ almost as an element of $\mathbb{P}_\gamma$ (it actually is if instead of all T^p we just project the elements in T^p that are permitted for p^γ). Therefore the tree T^p denotes all the possibile extensions for all the finite sequences p^γ, with γ in the support of p.

Definition 2.3. Let $p, q \in \mathbb{P}$. We say that p extends q and denote this by $p \leq q$ iff

1. $\mathrm{supp}(p) \supseteq \mathrm{supp}(q)$.

2. For every $\gamma \in \mathrm{supp}(q)$, p^γ is an end-extension of q^γ.

3. $p^{\mathrm{mc}(q)} \in T^q$.

4. For every $\gamma \in \mathrm{supp}(q)$,

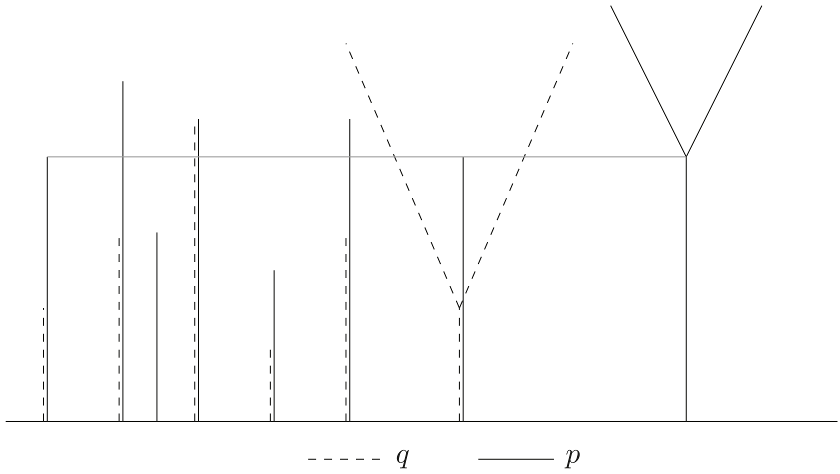
$$p^\gamma \setminus q^\gamma = \pi_{\mathrm{mc}(q),\gamma}[(p^{\mathrm{mc}(q)} \setminus q^{\mathrm{mc}(q)}) \restriction (\mathrm{lh}(p^{\mathrm{mc}(q)}) \setminus (i+1))],$$

where $i \in \mathrm{dom}(p^{\mathrm{mc}(q)})$ is the largest such that $p^{\mathrm{mc}(q)}(i)$ is not permitted for q^γ.

5. $\pi_{\mathrm{mc}(p),\mathrm{mc}(q)}$ projects $T^p_{p^{\mathrm{mc}\frown}}$ into $T^q_{q^{\mathrm{mc}\frown}}$.

6. For every $\gamma \in \mathrm{supp}(q)$ and $\mu \in \mathrm{Suc}_{T^p}(p^{\mathrm{mc}})$, if μ is permitted for p^γ, then $\pi_{\mathrm{mc}(p),\gamma}(\mu) = \pi_{\mathrm{mc}(q),\gamma}(\pi_{\mathrm{mc}(p),\mathrm{mc}(q)}(\mu))$.

It is something like this:

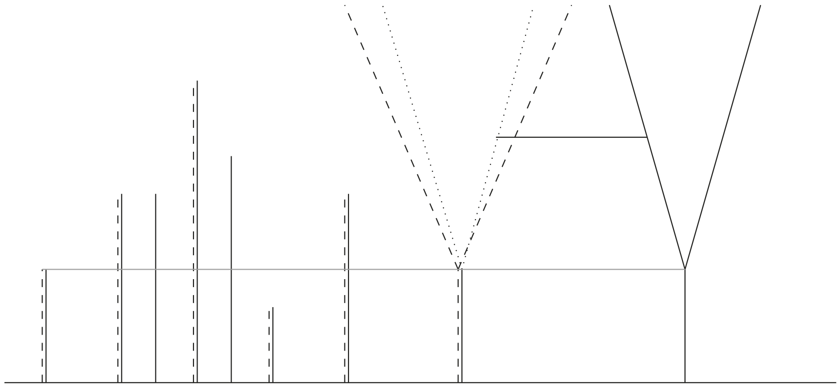


Definition 2.4. Let $p, q \in \mathbb{P}$. We say that p is a direct extension of q and denote this by $p \leq^* q$ iff

1. $p \leq q$

2. for every $\gamma \in \mathrm{supp}(q)$, $p^\gamma = q^\gamma$.

It is something like this:



Definition 2.5. Let $p \in \mathbb{P}$ and $t \in T^p_{p^{\mathrm{mc}\frown}}$. Then $p \oplus t$ is defined as follows:

1. $\mathrm{supp}(p \oplus t) = \mathrm{supp}(p)$;
2. $(p \oplus t)^{\mathrm{mc}} = p^{\mathrm{mc}\frown} t$;
3. $T^{p\oplus t} = \{s \in T^p : s \subseteq (p \oplus t)^{\mathrm{mc}} \vee (p \oplus t)^{\mathrm{mc}} \subseteq s\}$;
4. if $\gamma \in \mathrm{supp}(p)$,

$$(p \oplus t)^\gamma = p^{\gamma\frown}\pi_{\mathrm{mc}(p),\gamma}[t \restriction (\mathrm{lh}(t) \setminus (i_\gamma + 1))],$$

where i_γ is the largest such that $t(i)$ is not permitted by p^γ.

If s is a partial function from λ^{++} to the $^\circ$-increasing finite sequences in λ such that $|\mathrm{dom}(s)| < \lambda$, α is an ordinal such that $\alpha \geq_E \gamma$ for all $\gamma \in \mathrm{dom}(s)$ and t is a finite $^\circ$-increasing sequence of ordinals $< \lambda$, then $s \uplus (\alpha, t)$ is defined as follows:

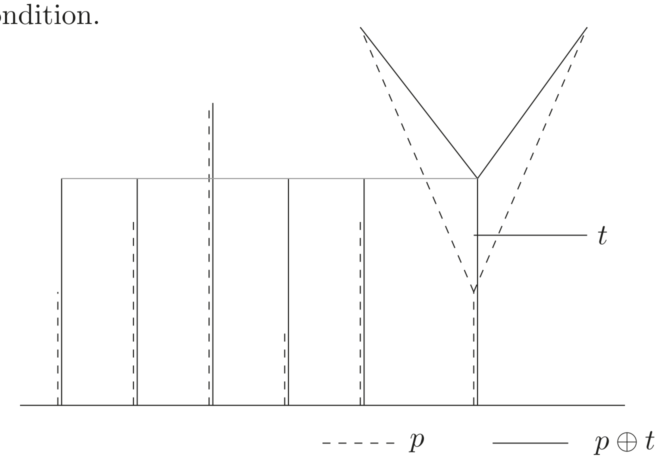
1. $\mathrm{dom}(s \uplus (\alpha, t)) = \mathrm{dom}(s)$;
2. if $\gamma \in \mathrm{dom}(s)$,

$$(s \uplus (\alpha, t))(\gamma) = s(\gamma)^\frown \pi_{\alpha,\gamma}[t \restriction (\mathrm{lh}(t) \setminus (i_\gamma + 1))],$$

where i_γ is the largest such that $t(i)$ is not permitted by $s(\gamma)$.

Note that $\mathrm{bas}(p \oplus t) = \mathrm{bas}(p) \uplus (\mathrm{mc}(p), p^{\mathrm{mc}\frown} t)$. For this reason, for every s as above and $\gamma \in \mathrm{dom}(s)$ we will write s^γ instead of $s(\gamma)$.

Morally, $p \oplus t$ is the weakest extension of p that we can have choosing t (and its projections) as extension: looking at Definition 2.3, we have three basic ways to extend t: extending the support (so adding new finite sequences, possibly shifting the maximum of the support), reducing T^p, or extending p^{mc}. Any combination of these three ways will produce an extension of p. If we use only the third way, then the "new" tree will have $p^{mc\frown}t$ as a trunk, and will consist of all the s such that $s \supseteq p^\frown t$ and, when possible, $p^{mc\frown}t$ will be projected to extend the finite sequences in the condition.



Theorem 2.6 (Gitik, Magidor)**.** *Let $\mathbb{P}$ be as above and suppose that GCH holds up to (and included) λ. Then*

$$V^{\mathbb{P}} \vDash 2^\lambda = \lambda^{++} \ \wedge \ \forall \kappa < \lambda \ 2^\kappa = \kappa^+$$

Proposition 2.7. *Let $\mathbb{P}$ be as above, and suppose that GCH holds up to (and included) λ. Then $\mathbb{P}$ has the *-Prikry condition.*

The rest of the paper is dedicated to prove Proposition 2.7.

The length measure for $\mathbb{P}$ will be $l(p) = \mathrm{lh}(p^{mc})$. So suppose that D is a dense open subset of $\mathbb{P}$ and that $p \in \mathbb{P}$. We need to prove that there are $n \in \omega$ and $q \leq^* p$ such that for any $r \leq q$ with $\mathrm{lh}(r^{mc}) = \mathrm{lh}(q^{mc}) + n$, we have that $r \in D$.

Without loss of generality, we can assume $p = \{\langle 0, \langle\rangle, TC\rangle\} = 1_{\mathbb{P}}$, where TC is the complete tree of the increasing finite sequences in λ, as the same construction will work for any $p \in \mathbb{P}$. Note that in this case any $q \in \mathbb{P}$ with $q^{\mathrm{mc}} = \langle\rangle$ is a direct extension of p.

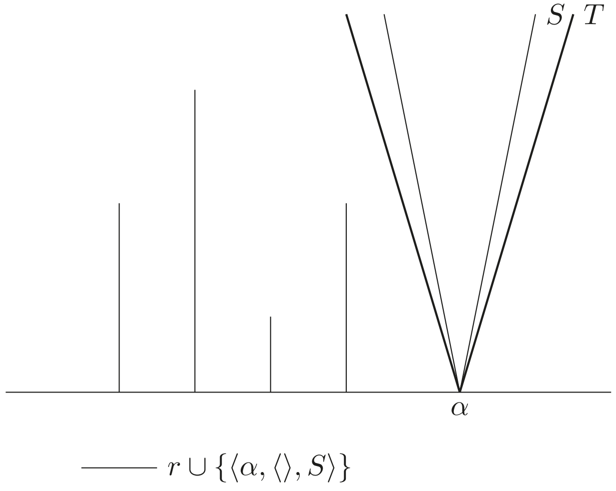
The proof goes through the same three steps as the proof for Lemma 1.7, but we are going to encounter a problem in the third step. Suppose we manage to prove the first two steps, so that we find a $q \leq^* 1_{\mathbb{P}}$ such that if there exist t and $r \in D$ so that $r \leq^* q \oplus t$, then $q \oplus t \in D$, and for any s_1, s_2 with the same length $q \oplus s_1 \in D$ iff $q \oplus s_2 \in D$. By density there exists $r \in D$ so that $r \leq q$, but it is not clear why there should exist a t such that $r \leq^* q \oplus t$ (this was immediate in the tree Prikry forcing), because possibly $\mathrm{mc}(r) > \mathrm{mc}(p)$. The solution is to fix an elementary submodel N of $H(\nu)$ with ν sufficiently large to contain all the relevant information, of cardinality λ^+ and closed under λ-sequences of its elements. This can be done since GCH holds up to λ. Then pick $\alpha < \lambda^{++}$ above all the elements of $N \cap \lambda^{++}$. Now we are going to consider only conditions p with $\mathrm{bas}(p), T^p \in N$ and $\mathrm{mc}(p) = \alpha$, and we prove the first two steps only for such conditions. Now, for this subset of conditions it is true that if $r \leq q$ there is a t such that $r \leq q \oplus t$, and since N is an elementary submodel this will be enough to prove the third step for the whole forcing.

Let T be a tree such that $\{\langle 0, \langle\rangle\rangle\} \cup \{\langle \alpha, \langle\rangle, T\rangle\}$ is in $\mathbb{P}$, with $T \in N$.

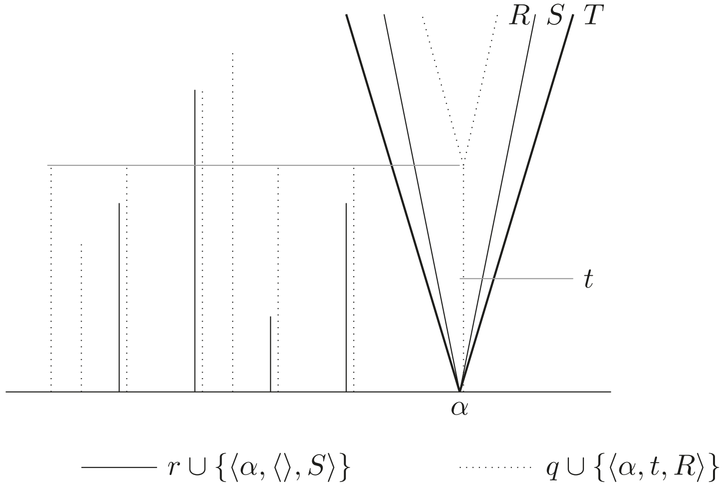
Lemma 2.8 (First step)**.** *There exists $r \cup \{\langle \alpha, \langle\rangle, S\rangle\} \in \mathbb{P}$, with $S \subseteq T$ and $r, S \in N$, such that for every $t \in S$, if for some $q, R \in N$, $q \cup \{\langle \alpha, t, R\rangle\} \leq^* (r \cup \{\langle \alpha, \langle\rangle, S\rangle\}) \oplus t$ and $q \cup \{\langle \alpha, t, R\rangle\} \in D$, then $(r \cup \{\langle \alpha, \langle\rangle, S\rangle\}) \oplus t \in D$.*

So, if there exists an extension q of $r \cup \{\langle \alpha, \langle\rangle, S\rangle\}$ with $\mathrm{bas}(q), S \in N$ and $\mathrm{mc}(q) = \alpha$ that is in D, then also the weakest extension q' of $r \cup \{\langle \alpha, \langle\rangle, S\rangle\}$ with $\mathrm{bas}(q') = \mathrm{bas}(q)$, $\mathrm{mc}(q') = \alpha$ and $q'^{\mathrm{mc}} = q^{\mathrm{mc}}$ is in D.

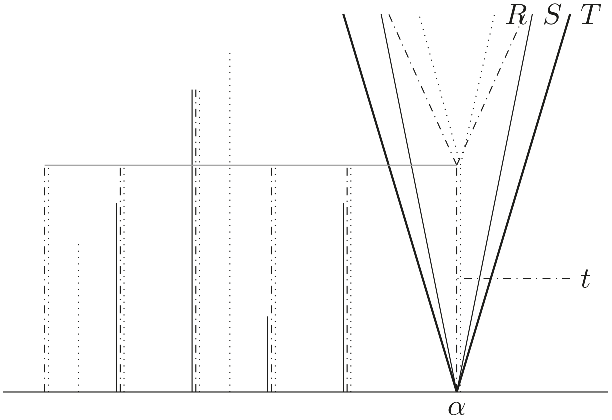
In other words, if there exists a condition like in the following picture (where the thicker line is $1_{\mathbb{P}}$):



such that if there exists something like the dotted condition in D



then the dash-dot condition is also in D:



——— $r \cup \{\langle \alpha, \langle\rangle, S\rangle\}$ -·-·-·- $(r \cup \{\langle \alpha, \langle\rangle, S\rangle\}) \oplus t$ ·········· $q \cup \{\langle \alpha, t, R\rangle\}$

Proof. If there is a $r \in N$ and a $T' \subseteq T$ with $T' \in N$ such that $r \cup \{\langle \alpha, \langle\rangle, T'\rangle\} \in D$, then this satisfies the Lemma.

If not, let $A = \mathrm{Suc}_T(\langle\rangle)$. We shall define by recursion the sequences $\langle r_\mu : \mu \in A\rangle$ and $\langle T^\mu : \mu \in A\rangle$, the first one increasing, such that $r_\mu \cup \{\langle \alpha, \langle\rangle, T^\mu\rangle\} \in \mathbb{P}$, and T^μ are in N.

Let $\mu = \min(A)$. If there are an $s \in N$ and a $T' \subseteq T$ in N with trunk $\langle \mu\rangle$ such that $s \cup \{\langle \alpha, \langle \mu\rangle, T'\rangle\} \in D$, then set $r_\mu = s$ and $T^\mu = T'$. Otherwise do nothing, i.e., $r_\mu = \{\langle 0, \langle\rangle\rangle\}$ and $T^\mu = T$.

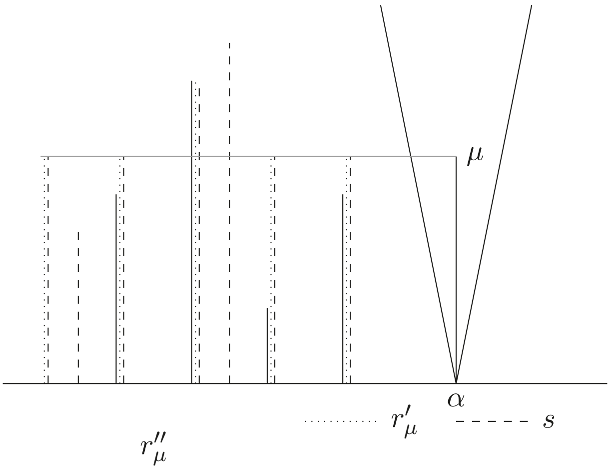
Suppose now that r_ξ and T^ξ are defined for any $\xi < \mu$ in A. Let $r''_\mu = \bigcup_{\xi \in \mu \cap A} r_\xi$ and consider $r'_\mu = r''_\mu \uplus (\alpha, \langle \mu\rangle)$. There are two cases:

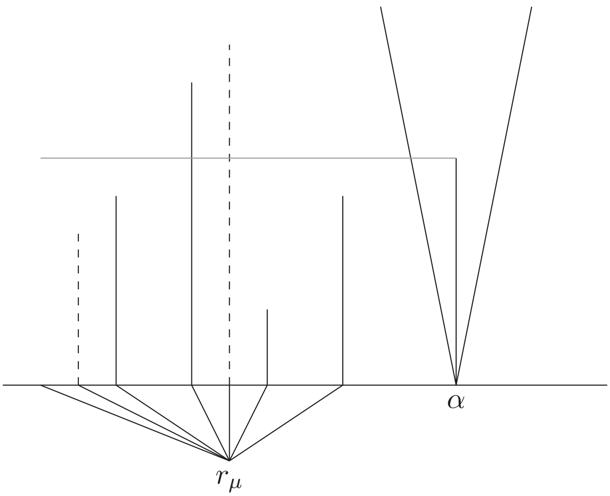
1. If there are an $s \in N$ and a $T' \subseteq T$ such that

$$D \ni s \cup \{\langle \alpha, \langle \mu\rangle, T'\rangle\} \leq^* r'_\mu \cup \{\langle \alpha, \langle \mu\rangle, T\rangle\},$$

   then set $r_\mu = r''_\mu \cup (s \restriction (\mathrm{dom}(s) \setminus \mathrm{dom}(r'_\mu)))$ and $T^\mu = T'$.

2. Otherwise do nothing, i.e., $r_\mu = r''_\mu$ and $T^\mu = T$.





Claim 2.9. *For any $\gamma \in \operatorname{dom}(r_\mu) \setminus \operatorname{dom}(r''_\mu)$, μ is not permitted for $(r_\mu)^\gamma$.*

Proof of Claim. If $\gamma \in \operatorname{dom}(r_\mu) \setminus \operatorname{dom}(r''_\mu)$ then we must be in the first case, so there was an $s \in N$ and a $T' \subseteq T$ as above such that $r_\mu = r''_\mu \cup (s \restriction (\operatorname{dom}(s) \setminus \operatorname{dom}(r'_\mu)))$. Then $\gamma \in \operatorname{dom}(s)$ and $s^\gamma = (r_\mu)^\gamma$.

But $s \cup \{\langle \alpha, \langle \mu \rangle, T' \rangle\} \in \mathbb{P}$, so by Definition 2.2(5) for any $\gamma \in \operatorname{dom}(s) \cup \{\alpha\}$, $\pi_{\alpha\gamma}(\max(\langle \mu \rangle)) = \pi_{\alpha\gamma}(\mu)$ is not permitted for $s^\gamma = (r_\mu)^\gamma$. By the full commutativity at λ the subclaim is proved. $\square$

Let $s_1 = \bigcup_{\mu \in A} r_\mu$. This will be the base of a condition in $\mathbb{P}$. We want now to define the tree of such condition, via the T^μ's, so some S^1 such that $s_1 \cup \{\langle \alpha, \langle\rangle, S^1 \rangle\}$ is an element of $\mathbb{P}$.

For $i < \lambda$ let

$$C_i = \begin{cases} A & \text{if there is no } \mu \in A \text{ such that } \mu^0 = i; \\ \bigcap_{\mu \in A, \mu^0 = i} \mathrm{Suc}_{T^\mu}(\langle \mu \rangle) & \text{otherwise.} \end{cases}$$

Note that $A \in U_\alpha$, and therefore we have that for any $i \in \lambda$, if there is a $\mu_1 \in A$ such that $\mu_1^0 = i$, for any $\mu_2 \in A$ with $\mu_2^0 > 9$, $|\{\mu \in A : \mu^0 = i\}| < \mu_2$, so by λ-completeness $C_i \in U_\alpha$. Set $A^* = A \cap \Delta^*_{i<\lambda} C_i = \{\nu \in A : \forall i < \nu^0 \ \nu \in C_i\} \in U_\alpha$. Then for every $\delta \in A^*$ and for every $\mu \in A$ if $\mu^0 < \delta^0$ then $\delta \in \mathrm{Suc}_{T^\mu}(\langle \mu \rangle)$. S^1 will be the tree obtained from T by eliminating all the branches that do not start with $\mu \in A^*$, replacing $T_{\langle \mu \rangle}$ with $T^\mu_{\langle \mu \rangle}$ and intersecting all the levels with A^*, i.e., $\langle \delta_0, \ldots, \delta_n \rangle \in S^1$ iff $\langle \delta_0, \ldots, \delta_n \rangle \in T^{\delta_0}$ and $\forall i \leq n$, $\delta_i \in A^*$.

Claim 2.10. *$s_1 \cup \{\alpha, \langle\rangle, S^1\} \in \mathbb{P}$.*

Proof of Claim. According to definition 2.2, to prove the claim we need $s_1 \cup \{\alpha, \langle\rangle, S^1\}$ to satisfy 6 conditions. Conditions (5) and (6) are trivial. Condition (1) holds because for any $\mu \in A$, $|r_\mu| \leq \lambda$, so s_1 is the countable union of sets of size $\leq \lambda$, and therefore $|s_1| \leq \lambda$. Condition (2) holds because $r_\mu \cup \{\langle \alpha, \langle\rangle, T^\mu \rangle \in \mathbb{P}$. We have just seen that S^1 satisfies Condition (3).

So the only non-trivial point is to show condition (4) of the definition of $\mathbb{P}$, i.e., that for any $\delta \in \mathrm{Suc}_{S^1}(\langle\rangle) = A^*$,

$$|\{\gamma \in \mathrm{dom}(s_1) : \delta \text{ is permitted for } (s_1)^\gamma\}| \leq \delta^0.$$

Let

$$B_\delta = \{\gamma \in \mathrm{dom}(s_1) : \delta \text{ is permitted for } (s_1)^\gamma\}.$$

Since $\mathrm{dom}(s_1) = \bigcup_{\mu \in A} \mathrm{dom}(r_\mu)$, we can divide B_δ in

$$B_{\delta,\mu} = \{\gamma \in \mathrm{dom}(r_\mu) : \delta \text{ is permitted for } (s_1)^\gamma = (r_\mu)^\gamma\}$$

with $\mu \in A$. We trace back the construction of s_1. For any $\mu \in A$, either $\mathrm{dom}(r_\mu) = \mathrm{dom}(r''_\mu) = \bigcup_{\xi \in A \cap \mu} \mathrm{dom}(r_\xi)$ (case (2)) or there is an s such that $\mathrm{dom}(r_\mu) = \bigcup_{\xi \in A \cap \mu} \mathrm{dom}(r_\xi) \cup \mathrm{dom}(s)$ (Case (1)). So $B_\delta = \bigcup \{B_{\delta,\mu} : \mu \in A, \ r_\mu \neq r''_\mu\}$.

By Subclaim 2.9 if μ is such that $r_\mu \neq r''_\mu$, then μ is not permitted for $(r_\mu)^\gamma$ for any $\gamma \in \mathrm{dom}(r_\mu) \setminus \bigcup_{\xi \in A \cap \mu} \mathrm{dom}(r_\xi)$. Note that in the construction of s_1 if $\mu' > \mu$ and $\gamma \in \mathrm{dom}(r_\mu)$, then $(r_\mu)^\gamma = (r_{\mu'})^\gamma$, therefore for any $\gamma \in \mathrm{dom}(r_\mu)$, $(s_1)^\gamma = (r_\mu)^\gamma$. So for any $\gamma \in \mathrm{dom}(r_\mu) \setminus \bigcup_{\xi \in A \cap \mu} \mathrm{dom}(r_\xi)$, μ is not permitted for $(s_1)^\gamma$. So if $\delta^0 < \mu^0$, also δ is not permitted for $(s_1)^\gamma$.

In other words, if $\delta^0 < \mu^0$ and $\delta \in B_{\delta,\mu}$, it must be that there is a $\xi < \mu$ such that $\delta \in B_{\delta,\xi}$. So we can write:

$$B_\delta = \bigcup \{B_{\delta,\mu} : \mu \in A,\ r_\mu \neq r''_\mu,\ \mu^0 < \delta^0\}.$$

But now, by the way we have chosen the U_α's, if $\mu^0 < \delta^0$ then $|\{\xi \in A : \xi^0 = \mu^0\}| < \delta_0$, therefore the former is a union of $\leq \delta^0$ elements.

Now fix a $B_{\delta,\mu}$, with $\mu \in A$, $r_\mu \neq r''_\mu$ and $\mu^0 < \delta^0$. Since $\delta \in A^*$, $\delta \in \mathrm{Suc}_{T^\mu}(\langle\mu\rangle)$ by definition of A^* and the fact that $\mu^0 < \delta^0$. But $r_\mu \cup \{\langle\alpha, \langle\mu\rangle, T^\mu\}$ is a condition in $\mathbb{P}$, therefore by point (4) of the definition of $\mathbb{P}$ we have

$$|\{\gamma \in \mathrm{dom}(r_\mu) : \delta \text{ is permitted for } (r_\mu)^\gamma\}| \leq \delta^0.$$

So, finally, for any $\mu \in A$, $r_\mu \neq r''_\mu$ and $\mu^0 < \delta^0$, $|B_{\delta,\mu}| \leq |\{\gamma \in \mathrm{dom}(r_\mu) : \delta \text{ is permitted for } (r_\mu)^\gamma\}| \leq \delta^0$. Therefore B_δ is the union of $\leq \delta_0$ sets of cardinality less then δ_0, so $|B_\delta| \leq \delta_0$. $\square$

Claim 2.11. *For every $\delta \in \mathrm{Suc}_{S^1}(\langle\rangle)$, if for some $q, R \in N$,*

$$q \cup \{\alpha, \langle\delta\rangle, R\} \leq^* (s_1 \cup \{\alpha, \langle\rangle, S^1\}) \oplus \langle\delta\rangle$$

and $q \cup \{\alpha, \langle\delta\rangle, R\} \in D$, then $(s_1 \cup \{\alpha, \langle\rangle, S^1\}) \oplus \langle\delta\rangle \in D$.

Proof of Claim. Recall the construction of s_1 at stage δ. We want to prove that $q \cup \{\alpha, \langle\delta\rangle, R\} \leq^* r'_\delta \cup \{\langle\alpha, \langle\delta\rangle, T\rangle\}$.

The only delicate point to prove is that for any $\gamma \in \mathrm{dom}(r'_\delta)$, $q^\gamma = (r'_\delta)^\gamma$, and we leave the rest of the (mostly trivial) details to the reader. By definition, $r'_\delta = r''_\delta \uplus (\alpha, \langle\delta\rangle)$, and $s_1 \upharpoonright \mathrm{dom}(r'_\delta) = r''_\delta$. Since

$$q \cup \{\alpha, \langle\delta\rangle, R\} \leq^* (s_1 \cup \{\alpha, \langle\rangle, S^1\}) \oplus \langle\delta\rangle$$

it must be that for any $\gamma \in \mathrm{dom}(s_1)$, so in particular for $\gamma \in \mathrm{dom}(r'_\delta)$,

$$q^\gamma = ((s_1 \cup \{\alpha, \langle\rangle, S^1\}) \oplus \langle\delta\rangle)^\gamma = (s_1 \uplus (\alpha, \langle\delta\rangle))^\gamma.$$

But for $\gamma \in \mathrm{dom}(r'_\delta)$ this is $(r''_\delta \uplus (\alpha, \langle\delta\rangle))^\gamma = (r'_\delta)^\gamma$.

But then this implies that in the definition of r_δ we are in Case (1), so there exists a s such that

$$s \cup \{\langle \alpha, \langle \delta \rangle, T^\delta \rangle\} \leq^* r'_\delta \cup \{\langle \alpha, \langle \mu \rangle, T \rangle\}$$

and $s \cup \{\langle \alpha, \langle \delta \rangle, T^\delta \rangle\} \in D$, with

$$(s_1)^\gamma = (r_\delta)^\gamma = \begin{cases} (r_\xi)^\gamma & \text{if there exists } \xi \in \delta \cap A,\ \gamma \in \operatorname{dom}(r_\xi); \\ s^\gamma & \text{otherwise.} \end{cases}$$

for any $\gamma \in \operatorname{dom}(r_\delta) = \operatorname{dom}(s)$. We want to prove now that

$$s_1 \cup \{\langle \alpha, \langle\rangle, S^1 \rangle\} \oplus \langle \delta \rangle \leq^* s \cup \{\langle \alpha, \langle \delta \rangle, T^\delta \rangle\}.$$

and by openness of D this suffices to prove the claim.

Since $S^1_{\langle \delta \rangle} \subseteq T^\delta_{\langle \delta \rangle}$, the only delicate point is again that for any $\gamma \in \operatorname{dom}(s) = \operatorname{dom}(r_\delta)$, $(s_1 \cup \{\langle \alpha, \langle\rangle, S^1 \rangle\} \oplus \langle \delta \rangle)^\gamma = s^\gamma$.

If $\gamma \in \operatorname{dom}(r_\delta)$, then by definition of r_δ either $\gamma \in \operatorname{dom}(r'_\delta) = \operatorname{dom}(r''_\delta)$, or $\gamma \in \operatorname{dom}(r_\delta) \setminus \operatorname{dom}(r''_\delta)$. Suppose that $\gamma \in \operatorname{dom}(r''_\delta)$. Then there exist a $\xi \in \delta \cap A$ such that $\gamma \in \operatorname{dom}(r_\xi)$, and $(s_1)^\gamma = (r_\xi)^\gamma$. But then $(s_1 \uplus (\alpha, \langle \delta \rangle))^\gamma = (r_\xi \uplus (\alpha, \langle \delta \rangle))^\gamma = (r'_\delta)^\gamma$. Since $s \cup \{\langle \alpha, \langle \delta \rangle, T^\delta \rangle\} \leq^* r'_\delta \cup \{\langle \alpha, \langle \mu \rangle, T \rangle\}$, $(r'_\delta)^\gamma = s^\gamma$, as we wanted.

Suppose then that $\gamma \in \operatorname{dom}(r_\delta) \setminus \operatorname{dom}(r''\delta)$. Then $(s_1)^\gamma = s^\gamma$, and since by Claim 2.9 δ is not permitted for $(s_1)^\gamma$, $(s_1 \uplus (\alpha, \langle \delta \rangle))^\gamma = (s_1)^\gamma = s^\gamma$, as we wanted. □

This was the first step, now we climb up the tree, by induction. We want to define $s_n, S^n \in N$ by induction on $n \in \omega$ such that $S^n \supseteq S^{n+1}$, $s_n \subseteq s_{n+1}$, $s_n \cup \{\langle \alpha, \langle\rangle, S^n \}\in \mathbb{P}$ and $S^{n+1} \upharpoonright \operatorname{lh}(n) = S^n$.

Suppose that s_n, S^n are already defined. We define r_t and T^t for any $t \in S^n$ of length $n+1$, by induction on the lexicographical order $\preccurlyeq$.

Let $r''_t = s_n \cup \bigcup_{s \preccurlyeq t} r_t$ and $r'_t = r''_t \uplus (\alpha, t)$. There are two cases:

1. If there are an $s \in N$ and a $T' \subseteq S^n$ such that

$$D \ni s \cup \{\langle \alpha, t, T' \rangle\} \leq^* r'_t \cup \{\langle \alpha, \langle\rangle, S^n \rangle\} \oplus t,$$

   then set $r_t = r''_t \cup s \upharpoonright (\operatorname{dom}(s) \setminus \operatorname{dom}(r'_t))$ and $T^t = T'$.

2. Otherwise do nothing, i.e., $r_t = r''_t$ and $T^t = S^n$.

Claim 2.12. *For any $\gamma \in \mathrm{dom}(r_t) \setminus \mathrm{dom}(r''_t)$, $t(n)$ is not permitted for $(r_t)^\gamma$.*

Proof of claim. As before. $\square$

Let $s_{n+1} = \bigcup_{t \in \mathrm{Lev}_n(S^n)} r_t$. For $i_0 < \cdots < i_n < \lambda$ let

$$C_{i_0,\dots,i_n} = \begin{cases} \{t(n) : t \in \mathrm{Lev}_n(S^n)\} & \text{if there is no } t \in \mathrm{Lev}_n(S^n) \text{ such that } \pi_{\alpha,0}{}''t = (i_0,\dots,i_n); \\ \bigcap_{t \in \mathrm{Lev}_n(S^n), \pi_{\alpha,0}{}''t = (i_0,\dots,i_n)} \mathrm{Suc}_{T^t}(t) & \text{otherwise.} \end{cases}$$

Claim 2.13. *For every $m \in \omega$, $i_0 < \cdots < i_m < \lambda$,*

$$|\{t \in \mathrm{Lev}_m(S^n) : \pi_{\alpha,0}{}''t = (i_0,\dots,i_m)\}| < \lambda.$$

Proof. We call

$$A_{i_0,\dots,i_m} = \{t \in \mathrm{Lev}_m(S^n) : \pi_{\alpha,0}{}''t = (i_0,\dots,i_m)\}.$$

The proof is by induction on $m \in \omega$. So fix $i_0 < \lambda$. Then

$$A_{i_0} = \{\langle\mu\rangle : \mu \in \mathrm{Suc}_{S^n}(\langle\rangle),\ \mu^0 = i_0\},$$

and this has cardinality $< \lambda$ since $\mathrm{Suc}_{S^n}(\langle\rangle) \in U_\alpha$ and by our choice of U_α.

Now consider $i_0 < \cdots < i_{m+1} < \lambda$. For any $u \in A_{i_0,\dots,i_m}$, consider $E(u) = \{\mu \in \mathrm{Suc}_{S^n}(u),\ \mu^0 = i_{m+1}\}$. Then, again, $|E(u)| < \lambda$. But by inductive hypothesis also $|A_{i_0,\dots,i_m}| < \lambda$, so $|A_{i_0,\dots,i_{m+1}}| = |\{u^\frown\langle\mu\rangle : u \in A_{i_0,\dots,i_m},\ \mu \in E(u)\}| < \lambda$. $\square$

Because of the claim, then, $C_{i_0,\dots,i_n} \in U_\alpha$. Now define $C^*_{i_0,\dots,i_{n-1}} = \triangle^*_{i<\lambda} C_{i_0,\dots,i_{n-1},i}$. Then for any $\delta \in C^*_{i_0,\dots,i_{n-1}}$, we have that if $t \in \mathrm{Lev}_n(S^n)$ is such that for any $j \leq n-1$ we have $t(j)^0 = i_j$, and $t(n)^0 < \delta^0$, then $\delta \in \mathrm{Suc}_{T^t}(t)$.

Define then by induction $C^*_{i_0,\dots,i_{n-2}} = \triangle^*_{i<\lambda} C^*_{i_0,\dots,i_{n-2},i}$, $C^*_{i_0,\dots,i_{n-3}} = \triangle^*_{i<\lambda} C^*_{i_0,\dots,i_{n-3},i}$, etc, until we have defined $C^* = \triangle^*_{i<\lambda} C^*_i$. Then, by definition, if $\delta \in C^*$ and $t \in \mathrm{Lev}_n(S^n)$ is such that for any $j \leq n$ $t(j)^0 < \delta^0$, then $\delta \in \mathrm{Suc}_{T^t}(t)$.

S^{n+1} will be the tree obtained from S^n by replacing T_t with T^t_t and intersecting all the levels with C^*, i.e., $\langle\delta_0,\dots,\delta_m\rangle \in S^{n+1}$ iff either $m \leq n$ and $\langle\delta_0,\dots,\delta_m\rangle \in S^n$, or $m > n$ and $\langle\delta_0,\dots,\delta_m\rangle \in T^{\delta_0,\dots,\delta_n}$, and moreover $\forall i \leq m$, $\delta_i \in C^*$.

Claim 2.14. $s_{n+1} \cup \{\alpha, \langle\rangle, S^{n+1}\} \in \mathbb{P}$.

Proof of Claim. The proof is similar to the previous one. In this case, we split B_δ in the union of

$$\{\gamma \in \mathrm{dom}(s_n) : \delta \text{ is permitted for } (s_{n+1})^\gamma = (s_n)^\gamma\}$$

and

$$B_{t,\delta} = \{\gamma \in \mathrm{dom}(r_t) : \delta \text{ is permitted for } (s_{n+1})^\gamma = (r_t)^\gamma\},$$

with $t(n)^0 < \delta^0$, because of Claim 2.12.

We are going to prove, by induction on $m \leq n$, that $|\{t \in \mathrm{Lev}_m(S^{n+1}) : t(m)^0 < \delta^0\}| \leq \delta^0$. Let

$$E(m) = \{t \in \mathrm{Lev}_m(S^{n+1}) : t(m)^0 < \delta^0\}.$$

For any $i < \delta^0$, since $\delta \in \mathrm{Suc}_{S^{n+1}}(\langle\rangle)$, $|\{\mu \in \mathrm{Suc}_{S^{n+1}}(\langle\rangle) : \mu^0 = i\}| < \delta^0$. Then

$$\begin{aligned}|E(0)| &= |\{\mu \in \mathrm{Suc}_{S^{n+1}}(\langle\rangle) : \mu^0 < \delta^0\}| = \\ &= |\bigcup_{i<\delta^0} \{\mu \in \mathrm{Suc}_{S^{n+1}}(\langle\rangle) : \mu^0 = i\}| \leq \delta^0.\end{aligned}$$

Fix now $u \in E(m)$ and $i < \delta^0$. Since δ, by definition, is in $\mathrm{Suc}_{S^{n+1}}(u)$ for any u such that $\pi_{\alpha,0}{}''u \subseteq \delta^0$, we have that $|\{\mu \in \mathrm{Suc}_{S^{n+1}}(u) : \mu^0 = i\}| < \delta^0$. But then

$$\begin{aligned}|E(m+1)| &= |\{t \in \mathrm{Lev}_m(S^{n+1}) : t(m)^0 < \delta^0\}| = \\ &= |\bigcup_{u\in E(m)} \bigcup_{i<\delta^0} \{u^\frown\langle\mu\rangle : \mu \in \mathrm{Suc}_{S^{n+1}}(u),\ \mu^0 = i\}| \leq \delta^0.\end{aligned}$$

Then $|\{t \in \mathrm{Lev}_n(S^{n+1}) : t(n)^0 < \delta^0\}| \leq \delta^0$, so there are only δ_0-many $B_{t,\delta}$. The proof that $|B_{t,\delta}| < \delta^0$ is the same as in claim 2.10, and so the claim is proved. $\square$

Claim 2.15. *For every $t \in S^{n+1}$, if for some $q, R \in N$,*

$$q \cup \{\alpha, t, R\} \leq^* (s_{n+1} \cup \{\alpha, \langle\rangle, S^{n+1}\}) \oplus t$$

and $q \cup \{\alpha, t, R\} \in D$, then $(s_{n+1} \cup \{\alpha, \langle\rangle, S^{n+1}\}) \oplus t \in D$.

Proof of Claim. As before. □

Finally, let $r = \bigcup_{n\in\omega} s_n$ and $S = \bigcap_{n\in\omega} S^n$. It is in easy calculation to show that $r \cup \{\langle \alpha, \langle\rangle, S\rangle\} \in \mathbb{P}$: for example, if $\delta \in \mathrm{Suc}_S(\langle\rangle)$, then $\delta \in \mathrm{Suc}_{S^n}(\langle\rangle)$ for any $n \in \omega$, therefore for any $n \in \omega$

$$|\{\gamma \in \mathrm{dom}(s_n) : \delta \text{ is permitted for } (s_n)^\gamma\}| \leq \delta^0.$$

But then $\{\gamma \in \mathrm{dom}(s) : \delta \text{ is permitted for } (s)^\gamma\}$ is the countable union of sets of size $\leq \delta^0$, and therefore it is itself of size $\leq \delta^0$.

Then $r \cup \{\langle \alpha, \langle\rangle, S\rangle\}$ is the condition we wanted. □

Lemma 2.16 (Second step)**.** *There exists $r \cup \{\langle \alpha, \langle\rangle, S^*\rangle\}$, with $S^* \subseteq S$, such that if $t_1, t_2 \in S$ are of the same length, then $(r \cup \{\langle \alpha, \langle\rangle, S^*\rangle\}) \oplus t_1 \in D$ iff $(r \cup \{\langle \alpha, \langle\rangle, S^*\rangle\}) \oplus t_2 \in D$.*

Proof. The r will be the same of the first step, so we work only on the tree S. The proof follows closely the proof of the second claim in Lemma 1.7, but it needs more care because now we require for $\eta_1 \geq_T \eta_2 \geq_T p^{mc}$, $\mathrm{Suc}_T(\eta_1) \subseteq \mathrm{Suc}_T(\eta_2)$. Therefore every time we reduce a level, we reduce also all the levels above, via an intersection. We are just going to prove the first step, as the rest will be the same as in Lemma 1.7.

Let

$$R = \{t \in S : r \cup \{\langle \alpha, \langle\rangle, S\rangle\} \oplus t \in D\}.$$

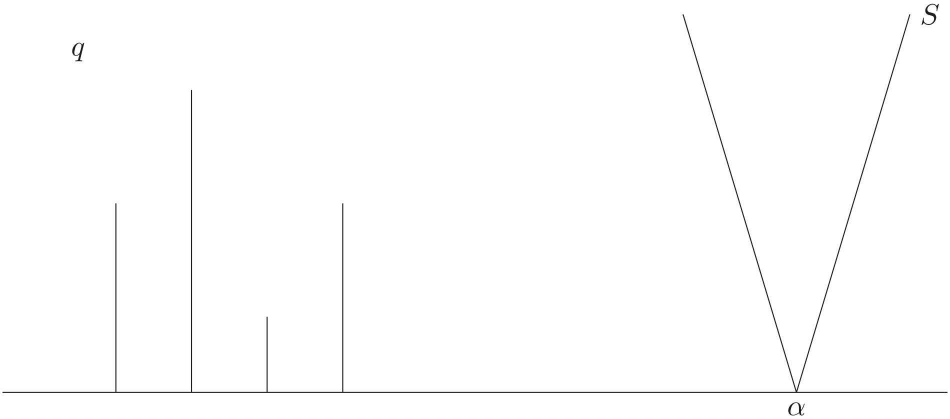
Therefore we have to find $S^* \subseteq S$ such that for any $t_1, t_2 \in S^*$ fo the same length, $t_1 \in R$ iff $t_2 \in R$.

Let $B^0_{\langle\rangle} = \{\delta \in \mathrm{Suc}_S(\langle\rangle) : t \in R\}$. Then exactly one among $B^0_{\langle\rangle}$ and $\mathrm{Suc}_S(\langle\rangle) \setminus B^0_{\langle\rangle}$ are in U_α. Call such $A^0_{\langle\rangle}$. Then let $\langle \mu_0, \dots, \mu_l\rangle \in S^0$ iff $\forall i\ \mu_i \in A^0_{\langle\rangle}$. We are intersecting all the levels of S to $A^0_{\langle\rangle}$ so that for any $\eta_1 \leq_{S^0} \eta_2$, $\mathrm{Suc}_{S^0}(\eta_2) \subseteq \mathrm{Suc}_{S^0}(\eta_1)$, and we are going to this this repeatedly without further comment. Note that for all $s \in S^0$, $\mathrm{Suc}_{S^0}(s) = \mathrm{Suc}_S(t) \cap A^0_{\langle\rangle}$, and the sequences in S^0 of length 1 are either all in R or all outside.

Now just follow the proof of Lemma 1.7, only $S^{n+1,0}$ is defined so that $\langle \mu_0, \dots, \mu_l\rangle \in S^{n+1,0}$ iff $\langle \mu_0, \dots, \mu_l\rangle \in S^n$ and $\forall i > n\ \mu_i \in A^{n+1}_{\langle \mu_0, \dots, \mu_n\rangle}$ (instead of just $\mu_{n+1} \in A^{n+1}_{\langle \mu_0, \dots, \mu_n\rangle}$), and the same change must be done in the definition of $S^{n,m}$. We leave the details to the reader. □

Lemma 2.17 (Third step)**.** *For any $p \in \mathbb{P}$ and any D dense set, there exist $q \leq^* p$ and $n \in \omega$ such that for any $t \in T^q$ with* $\mathrm{lh}(t) = n$*,* $q \oplus t \in D$*.*

Proof. We can suppose $p = 1_{\mathbb{P}}$. Like in Lemma 1.7, the first and second steps show that there is a $q = r \cup \{\langle \alpha, \langle\rangle, S\rangle\} \in \mathbb{P}$ such that if there are $s, t, R \in N$ such that $s \cup \{\langle \alpha, y, R\rangle\} \leq^* q \oplus t$ and $s \cup \{\langle \alpha, y, R\rangle\} \in D$, then for every $u \in S$ of the same length of t, $q \oplus u \in D$. This is our q.



We just need to prove that there are $s, t, R \in N$ such that

$$s \cup \{\langle \alpha, t, R\rangle\} \leq^* (r \cup \{\langle \alpha, \langle\rangle, S\rangle\}) \oplus t = q \oplus t$$

and $s \cup \{\langle \alpha, t, R\rangle\} \in D$, and then by definition of q the lemma is proved.

Pick some $\beta \in N \cap \lambda^{++}$ which is $\leq_E$ above every element of $\mathrm{dom}(r)$. This is possible since $\mathrm{dom}(r) \in N$. Note that $\beta \leq_E \alpha$ by the choice of α.

Claim 2.18. *There exists $S^* \subseteq S$ such that for every $\mu \in \mathrm{Suc}_{S^*}(\langle\rangle)$ and $\gamma \in \mathrm{dom}(r)$, if μ is permitted for r^γ, then $\pi_{\alpha,\gamma}(\mu) = \pi_{\beta,\gamma}(\pi_{\alpha,\beta}(\mu))$.*

Proof of claim. For any $\mu \in \mathrm{Suc}_S(\langle\rangle)$, let

$$B_\mu = \{\gamma \in \mathrm{dom}(r) : \mu \text{ is permitted for } r^\gamma\}.$$

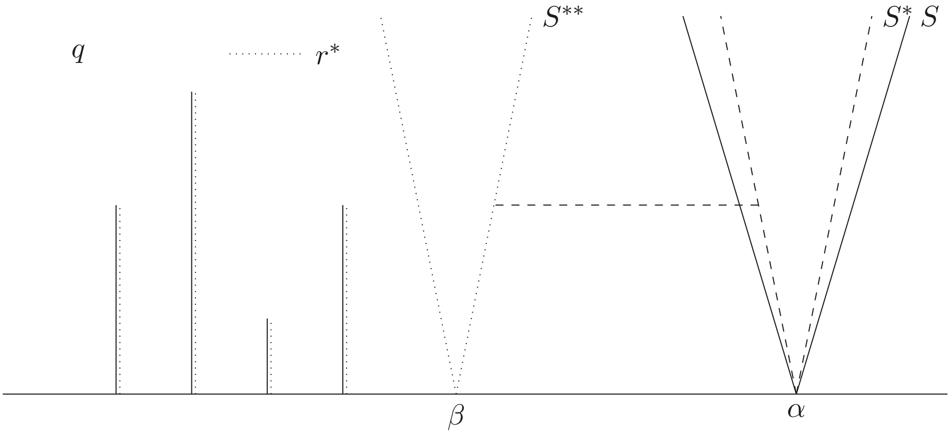
Then we have $|B_\mu| \leq \mu^0$. Let $\langle \xi_i : i < \lambda\rangle$ be an enumeration of $\mathrm{dom}(r)$ such that for any $\mu \in \mathrm{Suc}_S(\langle\rangle)$, $B_\mu \subseteq \{\xi_i : i < \mu^0\}$. For any $i < \lambda$, let

$$C_i = \{\mu \in \mathrm{Suc}_S(\langle\rangle) : \pi_{\alpha,\xi_i}(\mu) = \pi_{\beta,\xi_i}(\pi_{\alpha,\beta}(\mu))\}.$$

Let $A^* = \Delta^*_{i<\lambda} C_i$ and let S^* be the intersection of S with A^*. Then if $\mu \in \mathrm{Suc}_{S^*}(\langle\rangle)$, $\mu \in A^*$, so for all $i < \mu^0$ $\pi_{\alpha\xi_i}(\mu) = \pi_{\beta\xi_i}(\pi_{\alpha\beta}(\mu))$. But if μ

is permitted for r^γ, then $\gamma \in B_\mu$, so there should exist an $i < \mu^0$ such that $\gamma = \xi_i$, so $\pi_{\alpha\gamma}(\mu) = \pi_{\beta\gamma}(\pi_{\alpha\beta}(\mu))$. $\square$

Let S^{**} be the projection of S^* to β via $\pi_{\alpha,\beta}$. Let $r^* = r \cup \{\langle \beta, \langle\rangle, S^{**}\rangle\}$.



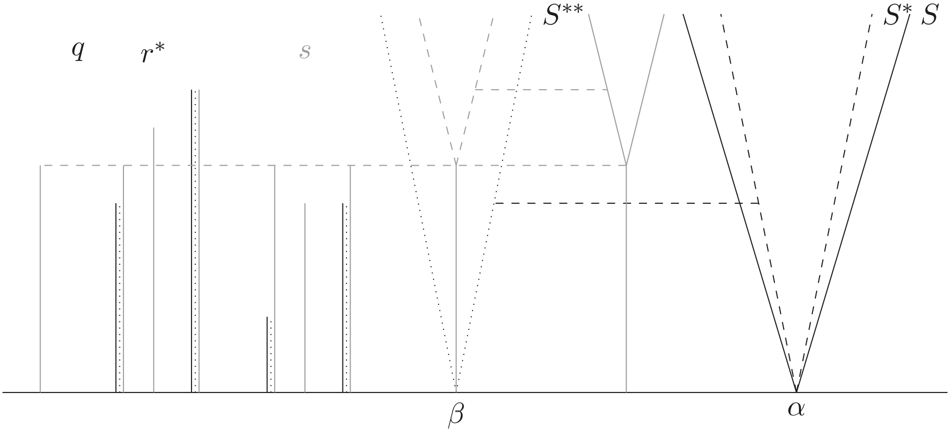
Claim 2.19. $r^* \in \mathbb{P}$.

Proof of claim. Again, the crux of the matter is to prove that for any $\delta \in \mathrm{Suc}_{S^{**}}(\langle\rangle)$,

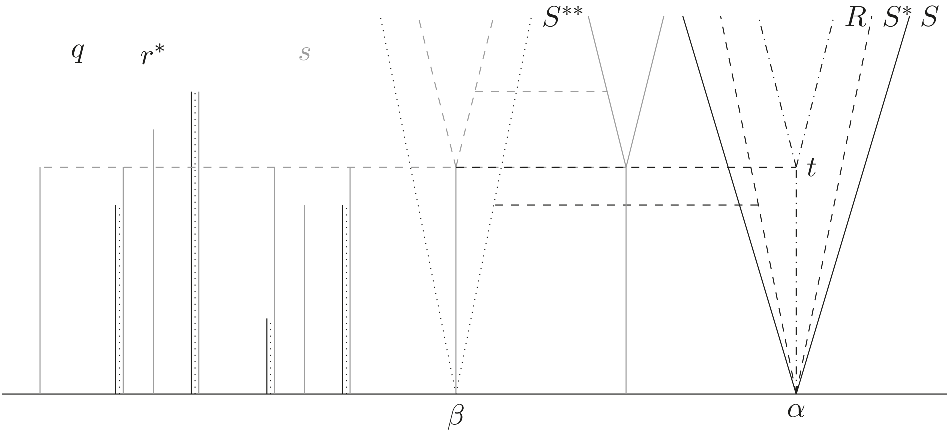
$$|\{\gamma \in \mathrm{dom}(r) : \delta \text{ is permitted for } r^\gamma\}| < \delta^0.$$

But if $\delta \in \mathrm{Suc}_{S^{**}}(\langle\rangle)$, then there exists $\mu \in \mathrm{Suc}_{S^*}(\langle\rangle)$ such that $\pi_{\alpha\beta}(\mu) = \delta$. But then by full commutativity $\mu^0 = \delta^0$, and since $r \cup \{\langle \alpha, \langle\rangle, S) \in \mathbb{P}\}$ and $S^* \subseteq S$ the claim is proved. $\square$

Then $r^* \in N \cap \mathbb{P}$, and since N is an elementary submodel there exists $s \in N \cap \mathbb{P}$, $s \leq r^*$ and $s \in D$.



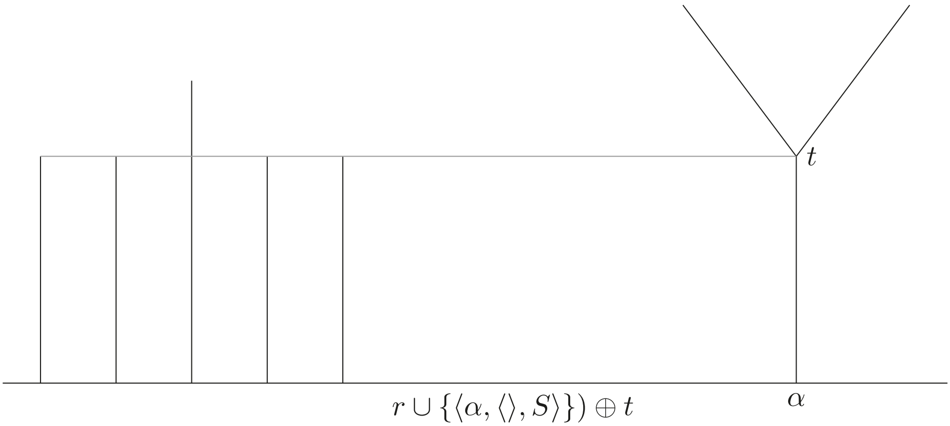
By definition of extension, $s^\beta \in S^{**}$, therefore there exists a $t \in S^*$ such that $\pi_{\alpha,\beta}{}''t = s^\beta$. Note also that $\mathrm{mc}(s) <_E \alpha$ by the choice of α. Let R be the tree with trunk t, derived intersecting S^*_t with $(\pi^{-1}_{\alpha,\mathrm{mc}(s)})''T^s$ and shrinking, if necessary, in order to insure the equality of projections $\pi_{\alpha,\gamma}$ and $\pi_{\mathrm{mc}(s),\gamma} \circ \pi_{\alpha,\mathrm{mc}(s)}$ for the relevant γ's in $\mathrm{dom}(s)$, as in claim 2.18.



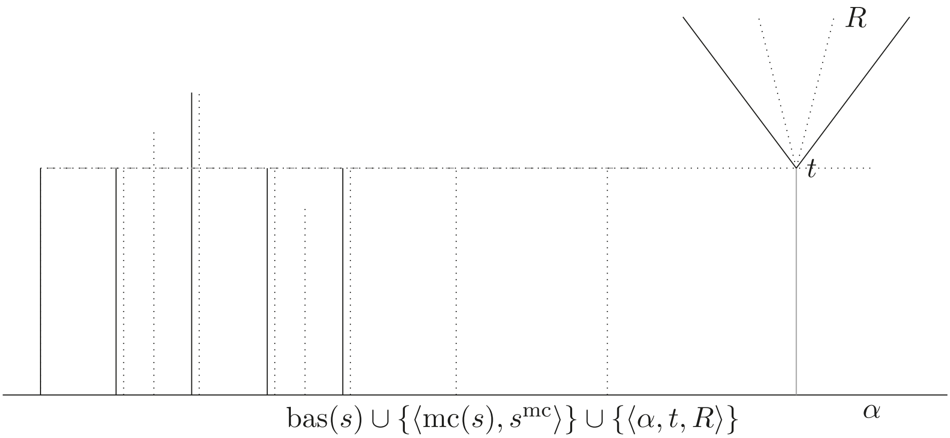
Then $\mathrm{bas}(s) \cup \{\langle \mathrm{mc}(s), s^{\mathrm{mc}}\rangle\} \cup \{\langle \alpha, t, R\rangle\} \leq s$, therefore it is in D.

Claim 2.20.

$$u = \mathrm{bas}(s) \cup \{\langle \mathrm{mc}(s), s^{\mathrm{mc}}\rangle\} \cup \{\langle \alpha, t, R\rangle\} \leq^* (r \cup \{\langle \alpha, \langle\rangle, S\rangle\}) \oplus t,$$



$r \cup \{\langle \alpha, \langle\rangle, S\rangle\}) \oplus t$



$\mathrm{bas}(s) \cup \{\langle \mathrm{mc}(s), s^{\mathrm{mc}}\rangle\} \cup \{\langle \alpha, t, R\rangle\}$

Proof of claim. Once again, we just prove that $u^\gamma = (r \uplus (\alpha, t))^\gamma$ for any $\gamma \in \mathrm{dom}(r)$, and we leave the rest to the reader.

First of all, u^γ is s^γ for any $\gamma \in \mathrm{supp}(s)$. Since $s \leq r^*$, for any $\gamma \in \mathrm{supp}(r^*)$, $s^\gamma = (r^*)^{\gamma\frown}\pi_{\beta\gamma}{}''s^\beta \restriction (\mathrm{lh}(s^\beta) \setminus (i+1))$, where i is the largest such that $s^\beta(i)$ is not permitted for $(r^*)^\gamma$.

On the other hand, for any $\gamma \in \mathrm{dom}(r)$ $(r \uplus (\alpha, t))^\gamma = r^{\gamma\frown}\pi_{\alpha\gamma}{}''t \restriction (\mathrm{lh}(t) \setminus (j+1))$, where j is the largest such that $t(j)$ is not permitted for $(r)^\gamma$.

Note that $\mathrm{dom}(r) = \mathrm{supp}(r^*) \setminus \{\alpha\}$, and for any $\gamma \in \mathrm{dom}(r)$, $r^\gamma = (r^*)^\gamma$. But now remember that $\pi_{\alpha\beta}{}''t = s^\beta$, $t \in S^*$ and by claim 2.18 $\pi_{\alpha\gamma}(\mu) = \pi_{\beta\gamma}(\pi_{\alpha\beta}(\mu))$ for $\mu \in \bigcup S^*$ and μ permitted for r^γ. Therefore

$$\pi_{\beta\gamma}{}''s^\beta \restriction (\mathrm{lh}(s^\beta) \setminus (i+1)) = \pi_{\alpha\gamma}{}''t \restriction (\mathrm{lh}(t) \setminus (j+1)),$$

and the claim is proved. $\square$

But then by the way we have chosen q we have that for every $v \in S$ of the same length of t, $q \oplus v \in D$, and the lemma is proved. $\square$

Finally, we can prove the *-Prikry condition for $\mathbb{P}$. Let q and n as in the last lemma. Let $r \leq q$ such that $l(r) = l(q) + n$. Let $t = \pi_{\mathrm{mc}(r)\alpha}{}'' r^{\mathrm{mc}}$. Since $r \leq q$, for any $\gamma \in \mathrm{supp}(q)$, r^γ is q^γ extended by the projection via $\pi_{\alpha\gamma}$ of r^α, that by definition is the projection via $\pi_{\mathrm{mc}(r)\alpha}$ of r^{mc}, that is t. But then $r^\gamma = (q \oplus t)^\gamma$, so $r \leq^* q \oplus t$, that is in D, so also $r \in D$.

In conclusion, *-Prikry condition is a topological property of forcing notions alternative to the Prikry condition, that seems to hold for many Prikry forcings, as the one above. Conditions in a Prikry forcing have two parts: a "sequence" part, usually a finite sequence or a set of finite sequences, that is an approximation of the generic we want, and a "future extension" part, usually a set of measure one or a tree, that establishes all the possible extensions of the "sequence" part. Therefore a way to prove the *-Prikry condition is to follow three steps: first trim the "future extension" part of the forcing (usually a set of measure one or a tree) so that if an extension of a certain length is in D, then all the extensions with the same "sequence" part of the first one are in D; then trim again so that all the extensions of the same length are either all in D or all not in D, and then argue in some way that there should be a condition that satisfy the Prikry sequence (this is immediate in simple forcings, but can be more complicated).

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