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MINIMAL SUBVARIETIES OF INVOLUTIVE RESIDUATED LATTICES

A b s t r a c t. It is known that classical logic $\mathbf{CL}$ is the single maximal consistent logic over intuitionistic logic $\mathbf{Int}$, which is moreover the single one even over the substructural logic $\mathbf{FL_{ew}}$. On the other hand, if we consider maximal consistent logics over a weaker logic, there may be uncountably many of them. Since the subvariety lattice of a given variety $\mathcal{V}$ of residuated lattices is dually isomorphic to the lattice of logics over the corresponding substructural logic $\mathbf{L}(\mathcal{V})$, the number of maximal consistent logics is equal to the number of minimal subvarieties[1] of the subvariety lattice of $\mathcal{V}$. Tsinakis and Wille have shown that there exist uncountably many atoms in the subvariety lattice of the variety of involutive residuated lattices. In the present paper, we will show that while there exist uncountably many atoms in the subvariety lattice of the variety of bounded representable involutive residuated lattices with mingle axiom $x^2 \leq x$, only two atoms exist in the subvariety lattice of the variety of bounded representable involutive residuated lattices with the idempotency $x = x^2$.

[1]For more information on minimal subvarieties, see Chapter 9 of [2]

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1 . Introduction

An algebra $\mathbf{A} = \langle \mathrm{A}, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ is a *residuated lattice* (RL) if $\mathbf{A}$ satisfies the following conditions.

(R1) $\langle \mathrm{A}, \wedge, \vee, 1\rangle$ is a lattice,

(R2) $\langle \mathrm{A}, \cdot, 1\rangle$ is a monoid with the unit 1,

(R3) for $x, y, z \in \mathrm{A}$, $x \cdot y \leq z \Leftrightarrow y \leq x\backslash z \Leftrightarrow x \leq z/y$.

(R3) is called the residuation law.

An RL $\mathbf{A}$ is *bounded* ($\mathrm{RL}_\perp$) if it has the greatest element $\top$ and least element $\perp$.

An RL $\mathbf{A}$ is *involutive* (InRL) if it has a constant 0, called involution constant, which satisfies the follwoing conditions:

1. $x\backslash 0 = 0/x$,
2. $0/(x\backslash 0) = (0/x)\backslash 0 = x$.

In InRL let us define a unary operation $'$ by $x' = x\backslash 0$. We call $'$ the involution.

An RL $\mathbf{A}$ is *representable* (RRL) if it can be represented as a subdirect product of totally ordered RLs.

A non-trivial algebra $\mathbf{A}$ is *strictly simple*, if it has neither non-trivial proper subalgebras nor non-trivial congruences. Note that the notion of proper subalgebras of an infinite algebra $\mathbf{A}$ is given as follows: A subalgebra $\mathbf{B}$ of $\mathbf{A}$ is *proper* if $\mathbf{B}$ is not isomorphic to $\mathbf{A}$. The fact that an algebra has no non-trivial proper subalgebras is enough to establish strict simplicity for a RL. For, congruences on residuated lattices correspond to convex normal subalgebras.

The bottom element $\perp \in \mathbf{A}$, when exists, is *nearly term-definable*, if there is an n-ary term-operation $t(\bar{x})$ such that for any n-tuple $\bar{a} \neq (\underbrace{1, \ldots, 1}_{n\text{-times}})$ of elements of A, $t(\bar{a}) = \perp$ holds.

A *variety* is a class of algebras which is closed under homomorphic images(H), subalgebras (S) and direct products (P). For any algebra $\mathbf{A}$, $\mathcal{V}(\mathbf{A}) = \mathsf{HSP}(\mathbf{A})$ is a variety generated by $\mathbf{A}$. Alternatively, it is an equational class, i.e. a class of the form $Mod(\{\mathcal{E}_i | i \in I\})$, where each $\mathcal{E}_i$ is an equation. A non-trivial variety $\mathcal{V}$ is called *minimal* if $\mathcal{V}$ has only trivial proper subvariety. We denote the variety of

InRL, $\mathrm{InRRL}_\perp$ with mingle axiom $x^2 \leq x$, and $\mathrm{InRRL}_\perp$ with idempotent axiom $x = x^2$, by $\mathcal{InRL}$, $\mathcal{InRRL}_\perp \cap Mod(x^2 \leq x)$ and $\mathcal{InRRL}_\perp \cap Mod(x = x^2)$ respectively. In the present paper, we discuss the number of minimal subvarieties of these varieties (see [5]). The following result, proved in [1], plays an important role when we show the minimality of a given variety.

Lemma 1. *Let* $\mathbf{A}$ *be a strictly simple* RL *with a nearly term definable bottom element* $\perp$*. Then,* $\mathcal{V}(\mathbf{A})$ *is a minimal variety.*

The next propositions show the numbers of minimal subvarieties of the variety of representable residuated lattices, and the variety of involutive residuated lattices, respectively.

Proposition 2.

1. *There are uncountably many minimal subvarieties of bounded representable residuated lattices with 3-potent axiom* $x^3 = x^4$ *([4]).*
2. *There are uncountably many minimal subvarieties of representable residuated lattices with idempotent axiom* $x = x^2$ *([1]).*

Proposition 3. *There are uncountably many minimal subvarieties of involutive residuated lattices ([5]).*

In the present paper, we demonstrate what will happen if these two conditions, i.e. representability and involutiveness, are combined. In Section 3, we show that the number of minimal subvarieties of bounded involutive representable residuated lattices even with mingle axiom is uncountable.

The situation changes radically when we replace the mingle axiom by idempotent axiom. In Section 4, we show that the number of minimal subvarieties of bounded involutive representable residuated lattices with idempotent axiom is only two.

2 . Adding involution

Here we give a construction of a bounded involutive RL from given upper-bounded RL, which is given by N. Galatos and J. G. Raftery (in [3]).

Let $\mathbf{A} = \langle \mathrm{A}, \wedge, \vee, \cdot, /, \backslash, 1 \rangle$ be an RL with the greatest element $\top$. Let $\mathrm{A}^- = \{a^- | a \in \mathrm{A}\}$ be a disjoint copy of A and $\mathrm{A}^* = \mathrm{A} \cup \mathrm{A}^-$. We extend the lattice order $\leq$ on A to A^* by stipulating that for any $a, b \in \mathrm{A}$,

1. $a^- < b$,
2. $a^- \leq b^- \leftrightarrow b \leq a$.

Thus, $\langle \mathrm{A}^*, \leq \rangle$ is order-isomorphic to the ordinal sum of the dual poset of $\langle \mathrm{A}, \leq \rangle$ and $\langle \mathrm{A}, \leq \rangle$ itself. Let $\bot = \top^-$ and $0 = 1^-$. Then $\bot$ is the least element of A^*. We define also a unary operation $'$ by $(a^-)' = a$ and $a' = a^-$ for any $a \in \mathrm{A}$. Then the operation $'$ satisfies the equation $x'' \approx x$. Therefore, we can identify $^-$ with $'$ by regarding each element $a \in \mathrm{A}$ as $(a^-)^-$.

Next we extend the monoid operation $\cdot$ on A to A^* as follows: For any $a, b \in \mathrm{A}$,

1. $a \cdot b' = (b/a)'$, $b' \cdot a = (a\backslash b)'$,
2. $a' \cdot b' = \bot$.

Finally, we extend the division operations $\backslash$ and $/$ on A to A^* as follows: For any $a, b \in \mathrm{A}$

1. $a\backslash b' = a'/b = (b \cdot a)'$,
2. $b'\backslash a = a/b' = \top$,
3. $a'\backslash b' = a/b$,
4. $b'/a' = b\backslash a$.

Then we can show that the operation $\cdot$ on A^* is a monoid operation for which residuation law holds with respect to $\backslash$ and $/$. Also, equations $x'' = x$ and $x\backslash y' = x'/y$ hold for $x, y \in \mathrm{A}^*$.

Lemma 4. *Let* $\mathbf{A}$ *be a member of the variety* $\mathcal{RRL}_\bot \cap Mod(x^2 \leq x)$. *Then* $\mathbf{A}^*$ *is also a member of the variety* $\mathcal{I}n\mathcal{RRL}_\bot \cap Mod(x^2 \leq x)$.

Proof. From the Galatos-Raftery construction we can show that $\mathbf{A}^*$ is an $\mathrm{InRRL}_\bot$. Moreover, for any $a \in \mathrm{A}$,

$$a^2 \leq a,$$
$$a'^2 = \bot \leq a'.$$

Thus $\mathbf{A}^*$ satisfies the mingle axiom. □

3 . Minimal subvarieties of $\mathcal{I}n\mathcal{RRL}_\perp \cap Mod(x^2 \leq x)$

In the remaining two sections, we discuss how many minimal subvarieties of $\mathcal{I}n\mathcal{RRL}_\perp$ exist for the case with the mingle axiom $x^2 \leq x$ and for the case with a stronger axiom $x^2 = x$.

In the following, we will define a bounded RL $\mathbf{D_S} = \langle \mathrm{D}, \wedge, \vee, \cdot_\mathrm{S}, \backslash_\mathrm{S}, /_\mathrm{S}, 1 \rangle$ with mingle axiom, for each subset S of natural numbers $\mathbb{N}$. Let us define a set D by

$$\mathrm{D} = \{a_i | i \in \mathbb{N}^+\} \cup \{b_i | i \in \mathbb{N}\} \cup \{1\}$$

where $\mathbb{N}^+$ is the set of positive integers. We define an order $\leq$ on D as follows:

$$b_0 < b_i \leq b_j \leq 1 \leq a_k \leq a_l$$
$$\Leftrightarrow \text{ for all } i, j, k, l \in \mathbb{N}^+, i \leq j \text{ and } k \geq l.$$

Obviously, the order $\leq$ is total (see Figure 1). For a given subset S of $\mathbb{N}$, we define a multiplication $\cdot_\mathrm{S}$ on $\mathbf{D}$ depending of S by

$$x \cdot_\mathrm{S} 1 = 1 \cdot_\mathrm{S} x = x \quad \text{for every } x \in \mathrm{D}$$

$$a_i \cdot_\mathrm{S} a_j = a_{min\{i,j\}}$$

$$b_i \cdot_\mathrm{S} b_j = b_{min\{i,j\}}$$

$$b_j \cdot_\mathrm{S} a_i = \begin{cases} b_j & \text{if } j < i \text{ or } i = j \in S \\ a_i & \text{if } i < j \text{ or } i = j \notin S \end{cases}$$

$$a_i \cdot_\mathrm{S} b_j = \begin{cases} a_i & \text{if } i < j \text{ or } i = j \in S \\ b_j & \text{if } j < i \text{ or } i = j \notin S \end{cases}$$

It is easy to see that our multiplication is associative. Next, we define two division operations by

$$x \backslash_\mathrm{S} y = \bigvee \{z | x \cdot_\mathrm{S} z \leq y\},$$
$$y /_\mathrm{S} x = \bigvee \{z | z \cdot_\mathrm{S} x \leq y\}.$$

Note that the right-hand sides of both of the above equations always exist, since the lattice-reduct of $\mathbf{D}_S$ is complete. Moreover, the residuation law holds between $\cdot_\mathrm{S}$ and $\backslash_\mathrm{S}$ ($/_\mathrm{S}$). Thus, $\mathbf{D_S}$ is a bounded RL in which a_1 is the top element and b_0 is the bottom element. Moreover it satisfies mingle axiom as $x \cdot_\mathrm{S} x = x$.

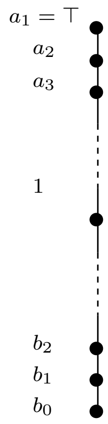

Figure 1. The residuated lattice $\mathbf{D}_S$

We construct an InRL from this algebra $\mathbf{D}_S$ by the Galatos-Raftery construction, mentioned in the previous section. Then we get a bounded InRL $\mathbf{D}^*_\mathbf{S}$ with mingle axiom for each subset S of natural numbers by Lemma 4. Now, we will show the following.

Theorem 5. *There are uncountably many minimal subvarieties of bounded involutive residuated lattices with mingle axiom.*

Proof. It is enough to prove the following:

1. For any $S \subseteq \mathbb{N}$, $\mathbf{D}^*_\mathbf{S}$ is a strictly simple algebra.
2. The element $\bot \in \mathbf{D}^*_\mathbf{S}$ is nearly term definable lower bound.
3. If S_1 and S_2 are distinct subset of $\mathbb{N}$ then $\mathbf{D}^*_{\mathbf{S_1}}$ and $\mathbf{D}^*_{\mathbf{S_2}}$ generate distinct varieties.

To prove that $\mathbf{D}^*_\mathbf{S}$ is strictly simple, it suffices to show that $\mathbf{D}^*_\mathbf{S}$ is generated by 1. Obviously, $0 = 1'$ and $0\backslash 1 = \top$. For each $w = 1, 2$, we have

if $i \in S_w$ then $1/a_i = b_i$ and $1/b_i = a_{i+1}$,

if $i \notin S_w$ then $a_i\backslash 1 = b_i$ and $b_i\backslash 1 = a_{i+1}$,

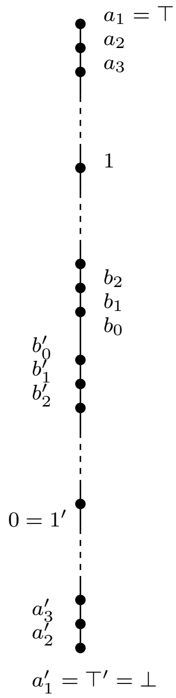

Figure 2. The InRL $\mathbf{D}_S^*$

and $(1/a_1) \wedge (a_1\backslash 1) = b_0$. Thus we can generate all elements of $\mathbf{D_S}$ inductively. Finally, we can get ${a_i}'$ and ${b_i}'$ by

$$a_i\backslash 0 = {a_i}' \text{ and } b_i\backslash 0 = {b_i}'.$$

Hence $\mathbf{D_S}^*$ is strictly simple.

Next, let us define a term $q_\bot(x)$ as follows:

$$q_\bot(x) = (x \wedge x')^2.$$

Suppose that $x \neq 1$. If $x \in \mathrm{D_S}$ then $x > x' \in \mathrm{D_S}'$. If $x \in \mathrm{D_S}'$ then $x < x' \in \mathrm{D_S}$. Hence $x \wedge x' \in \mathrm{D}'_S$ for any $x \neq 1$, and thus $(x \wedge x')^2 = \bot$. Therefore $\bot$ is nearly term-definable lower bound.

Now we show that for any pair of distinct sets $S_1, S_2 \in \mathbb{N}$, $\mathcal{V}(D_{S_1})$ and $\mathcal{V}(D_{S_2})$ generate distinct varieties. We define terms t_a, t_b and t as follows:

$t_a(x) \approx (1/x) \wedge (x\backslash 1),$
$t_b(x) \approx (1/x) \vee x(\backslash 1),$
$t(x) \approx t_a(t_b(x)).$

Suppose that S_1 and S_2 are distinct sets. Without loss of generality, we may assume that there exists $i \in \mathbb{N}^+$ such that $i \in S_1$ and $i \notin S_2$. Then $b_i \cdot_{S_1} a_i = b_i$ but $b_i \cdot_{S_2} a_i = a_i$. Now we define constant terms q_{b_i} and q_{a_i} by

$q_{b_i} \approx t_b(t^{i-1}(1'\backslash 1))$
$q_{a_i} \approx t^{i-1}(1'\backslash 1).$

The equation $q_{b_i} \cdot q_{a_i} \approx q_{b_i}$ holds in $\mathbf{D}^*_{\mathbf{S_1}}$ but not in $\mathbf{D}^*_{\mathbf{S_2}}$. So $\mathcal{V}(\mathbf{D}^*_{\mathbf{S_1}})$ satisfies the equation $q_{b_i} \cdot q_{a_i} \approx q_{b_i}$, but $\mathcal{V}(\mathbf{D}^*_{\mathbf{S_2}})$ does not satisfies it. Hence $\mathcal{V}(\mathbf{D}^*_{\mathbf{S_1}}) \neq \mathcal{V}(\mathbf{D}^*_{\mathbf{S_2}})$. □

4 . Minimal subvarieties of $\mathcal{I}n\mathcal{RRL}_\perp \cap Mod(x = x^2)$

In the previous section we show that $\mathcal{I}n\mathcal{RRL}_\perp \cap Mod(x^2 \leq x)$ has uncountably many atoms. In contrast with this, the number of minimal subvarieties of bounded representable $\mathcal{I}n\mathcal{RRL}_\perp \cap Mod(x = x^2)$ is only two, as we show below.

First we define three $\mathrm{InRRL}_\perp$s $\mathbf{2}$, $\mathbf{3}$ and $\mathbf{4}$ with idempotent axiom $x = x^2$ as follows:

$\mathbf{2} = \langle 2, \wedge_\mathbf{2}, \vee_\mathbf{2}, \cdot_\mathbf{2}, /_\mathbf{2}, \backslash_\mathbf{2}, 1, 0, 1\rangle,$
$\mathbf{3} = \langle 3, \wedge_\mathbf{3}, \vee_\mathbf{3}, \cdot_\mathbf{3}, /_\mathbf{3}, \backslash_\mathbf{3}, 1, \perp, \top\rangle,$
$\mathbf{4} = \langle 4, \wedge_\mathbf{4}, \vee_\mathbf{4}, \cdot_\mathbf{4}, /_\mathbf{4}, \backslash_\mathbf{4}, 1, \perp, \top\rangle,$

where sets 2, 3 and 4 are underlying sets defined by $2 = \{0, 1\}$, $3 = \{\perp, 1, \top\}$, $4 = \{\perp, 0, 1, \top\}$, respectively. We define orders on 2, 3 and 4 by

$0 \leq 1,$
$\perp \leq 1 \leq \top,$
$\perp \leq 0 \leq 1 \leq \top.$

We define also monoid operations on $\mathbf{2}$, $\mathbf{3}$ and $\mathbf{4}$ by the following tables.

$\cdot_2$	1	0
1	1	0
0	0	0

$\cdot_3$	$\top$	1	$\bot$
$\top$	$\top$	$\top$	$\bot$
1	$\top$	1	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$

$\cdot_4$	$\top$	1	0	$\bot$
$\top$	$\top$	$\top$	$\top$	$\bot$
1	$\top$	1	0	$\bot$
0	$\top$	0	0	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

Involution is defined by $1' = 0$, $0' = 1$, $\top' = \bot$ and $\bot' = \top$ in all of these algebras. The residuation law holds in all of $\mathbf{2}$, $\mathbf{3}$ and $\mathbf{4}$. Thus they are bounded involutive representable residuated lattices with idempotent axiom.

By using these algebras, we can show the following theorem.

Theorem 6. *There exist only two minimal subvarieties of bounded involutive representable residuated lattices with idempotent axiom.*

Proof. First we show that any subdirectly irreducible $\mathbf{A} \in \mathcal{I}n\mathcal{RRL}_\bot + (x = x^2)$ has a subalgebra which is isomorphic to one of $\mathbf{2}$, $\mathbf{3}$ and $\mathbf{4}$. Since $\mathbf{A}$ satisfies idempotent axiom we can show $0 \leq 1$. Also, it is easy to see that $\bot = 0$ iff $\top = 1$. Suppose that $\mathbf{A}$ satisfies $0 = 1$. Clearly $\{\bot, 1, \top\} \subseteq \mathrm{A}$ and it is closed under monoid operation and involution. Moreover $\top\backslash 1 = (\top\backslash 1')'' = (\top 1)' = \top' = \bot$ hold. By using this we can show that $\{\bot, 1, \top\}$ is closed under residuation. Hence $\{\bot, 1, \top\}$ is a subalgebra of $\mathbf{A}$ which is isomorphic to $\mathbf{3}$.

Suppose next that $\mathbf{A}$ satisfies $0 < 1$ and $\top = 1$. Then 1 is the greatest and 0 is the least element of $\mathbf{A}$. Clearly $\{0, 1\} \subseteq \mathrm{A}$ and it is closed under monoid operation, residuation and involution. Hence $\{0, 1\}$ is a subalgebra of $\mathbf{A}$ which is isomorphic to $\mathbf{2}$.

Finally suppose that $\mathbf{A}$ satisfies $0 < 1$ and $\top \neq 1$. We have $\bot \neq 0$. Clearly $\{\bot, 0, 1, \top\} \subseteq \mathrm{A}$ and it is closed under involution. Let $0\backslash\bot = x$. If $x \geq 0$ then $0 = 0^2 \leq 0 \cdot x = \bot$. This is a contradiction. Thus $x < 0$. Then $x = x^2 \leq x \cdot 0 = \bot$. Therefore $0\backslash\bot = \bot$. Since $\mathbf{A}$ is involutive, we have $\top \cdot 0 = \top$. Hence $\{\bot, 0, 1, \top\}$ is closed under monoid operation. We can also show that it is closed under residuation. Hence $\{\bot, 0, 1, \top\}$ is a subalgebra of $\mathbf{A}$ which is isomorphic to $\mathbf{4}$.

On the other hand, we show that the algebra $\mathbf{3}$ is a homomorphic image of $\mathbf{4}$. In fact, the map f defined by $f(\top) = \top$, $f(1) = f(0) = 1$ and $f(\bot) = \bot$ gives such a homomorphism. So $\mathbf{3}$ is an element of the subvariety generated by $\mathbf{4}$. It is easy to see that $\mathbf{2}$ and $\mathbf{3}$ have no proper subalgebras. Therefore, only $\mathcal{V}(\mathbf{2})$ and $\mathcal{V}(\mathbf{3})$ are minimal subvarieties of $\mathcal{I}n\mathcal{RRL}_\bot \cap Mod(x = x^2)$. Note that the InRL $\mathbf{2}$ is essentially equivalent to the two-element Boolean algebra. $\square$

5 . Logical consequences

In this section we show what is the meaning of our theorems from a logical point of view. We introduce the logic $\mathbf{InFL}'$ which corresponds to variety of involutive residuated lattices. Our language consists of $\wedge, \vee, \cdot, \backslash, /, \neg$ as logical connectives, and of 1, $\top$ and $\bot$ as logical constants. The logic $\mathbf{InFL}'$ is introduced as a sequent calculus obtained from $\mathbf{FL}$ by deleting both an initial sequent and an inference rule for the logical constant 0. Moreover we add the following initial sequent and inference rules:

$$\neg\neg\alpha \Rightarrow \alpha,$$

$$\frac{\alpha, \Gamma \Rightarrow}{\Gamma \Rightarrow \neg\alpha}\ (\Rightarrow \neg) \quad \frac{\Gamma \Rightarrow \alpha}{\neg\alpha, \Gamma \Rightarrow}\ (\neg \Rightarrow) \quad \frac{\Sigma, \Gamma \Rightarrow}{\Gamma, \Sigma \Rightarrow}\ (cycling).$$

We can show the following lemma.

Lemma 7. *(1)* $L(\mathcal{InRL}) = \mathbf{InFL}'$. *(2)* $V(\mathbf{InFL}') = \mathcal{InRL}$.

Note that the logic $\mathbf{InFL}' + exchange$ corresponds to the logic $\mathbf{InFL_e}$ since $\neg 1$ is defined by $1 \to 0 (= 0)$ in $\mathbf{FL_e}$.

Next we give an axiomatization of the logic determined by $\mathcal{RRL}$ and $\mathcal{RL}_\bot$ respectively. The variety $\mathcal{RRL}$ is axiomatized by

$$\lambda_z((x \vee y)/x) \vee \rho_w((x \vee y)/y) \equiv 1.$$

Thus to get the sequent calculus of the logic determined by the variety $\mathcal{RRL}$, we need to add

(R) $\Rightarrow \lambda_\alpha((\varphi \vee \psi)/\varphi) \vee \rho_\beta((\varphi \vee \psi)/\psi)$

as initial sequents. Here λ_z and ρ_w are left conjugate and right conjugate, respectively, and λ_α and ρ_β are formulas corresponding to conjugates.

To get the sequent calculus of the logic determined by the variety $\mathcal{RL}_\bot$, we need moreover the following initial sequents:

(T) $\Gamma \Rightarrow \top$,
(B) $\Gamma, \bot, \Delta \Rightarrow \gamma$.

From a logical point of view, our theorems in Section 3 and 4 have the following meaning.

Corollary 8. *1. There are uncountably many maximal consistent logics over the logic* $\mathbf{InFL}'+$ *(R)* $+$ *(T)* $+$ *(B)* $+(\alpha \cdot \alpha \Rightarrow \alpha)$.

2. On the other hand, there exists a single maximal consistent logics over $\mathbf{InFL}'+$ *(R)* $+$ *(T)* $+$ *(B)* $+(\alpha \cdot \alpha \Rightarrow \alpha) + (\alpha \Rightarrow \alpha \cdot \alpha)$, *except the classical logic.*

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References

[1] N. Galatos, *Minimal varieties of residuated lattices*, Algebra Universalis **52** (2005), pp. 215–239.

[2] N.Galatos, P.Jipsen, T. Kowalski and H.Ono, Residuated Lattices: an algebraic glimpse at substructural logics, Studies in Logic and the Foundations of Mathematics, vol. 151, Elsevier Science, 2007.

[3] N. Galatos and J. G. Raftery, *Adding involution to residuated structures*, Studia Logica **77** (2004), pp. 181–207.

[4] P. Jipsen and C. Tsinakis, *A survey of residuated lattices*, Ordered Algebraic Structures, Kluwer, Dordrecht, 2002, pp. 19–56.

[5] C. Tsinakis and A. M. Wille, *Minimal varieties of involutive residuated lattices*, Studia Logica **83** (2006), pp. 407–423.

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