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# THE VARIETY OF SEMI-HEYTING ALGEBRAS SATISFYING THE EQUATION $(0 \to 1)^* \vee (0 \to 1)^{**} \approx 1$

A b s t r a c t. In [4, Definition 8.1], some important subvarieties of the variety $\mathcal{SH}$ of semi-Heyting algebras are defined. The purpose of this paper is to introduce and investigate the subvariety $\mathcal{ISSH}$ of $\mathcal{SH}$, characterized by the identity $(0 \to 1)^* \vee (0 \to 1)^{**} \approx 1$. We prove that $\mathcal{ISSH}$ contains all the subvarieties introduced by Sankappanavar and it is in fact the least subvariety of $\mathcal{SH}$ with this property. We also determine the sublattice generated by the subvarieties introduced in [4, Definition 8.1] within the lattice of subvarieties of semi-Heyting algebras.

[1] I wish to dedicate this work to my father Francisco Cornejo.
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## 1. Introduction and preliminaries

In [4], Sankappanavar introduced a new equational class $\mathcal{SH}$ of algebras, which he called "*Semi-Heyting Algebras*", as an abstraction of Heyting algebras. This variety includes Heyting algebras and share with them some rather strong properties. For example, the variety of semi-Heyting algebras is arithmetical, semi-Heyting algebras are pseudocomplemented distributive lattices and their congruences are determined by filters. Sankappanavar introduced in his work several subvarieties of $\mathcal{SH}$, for instance, the variety $\mathcal{SH}^S$ of Stone semi-Heyting algebras, the variety $\mathcal{SH}^B$ of Boolean semi-Heyting algebras, the variety $\mathcal{QH}$ of quasi-Heyting algebras, the variety $\mathcal{SH}^C$ generated by semi-Heyting chains, investigated in [1], the variety $\mathcal{FTT}$ in which $0 \to 1 \approx 1$, the variety $\mathcal{FTF}$ in which $0 \to 1 \approx 0$, and so on. These new varieties seem to be of interest from the point of view of non-classical logic, since they can provide a new interpretation for the implication connective.

The purpose of this paper is to introduce and investigate the subvariety $\mathcal{ISSH}$ of semi-Heyting algebras satisfying the equation $(0 \to 1)^* \vee (0 \to 1)^{**} \approx 1$. Clearly, the variety of Stone semi-Heyting algebras is contained in $\mathcal{ISSH}$. Moreover, $\mathcal{ISSH}$ contains all the subvarieties introduced in [4], and it is in fact the least subvariety of $\mathcal{SH}$ that contains all the subvarieties of Sankappanavar.

We start by recalling some definitions and basic results ([2], [3] and [4]).

A semi-Heyting algebra is an algebra $\mathbf{L} = \langle L, \vee, \wedge, \to, 0, 1\rangle$ such that

(SH1) $\langle L, \vee, \wedge, 0, 1\rangle$ is a lattice with 0 and 1,

(SH2) $x \wedge (x \to y) \approx x \wedge y$,

(SH3) $x \wedge (y \to z) \approx x \wedge [(x \wedge y) \to (x \wedge z)]$,

(SH4) $x \to x \approx 1$.

Semi-Heyting algebras are pseudocomplemented distributive lattices, with the pseudocomplement given by $x^* = x \to 0$ (see [4]). Nevertheless, the operation $\to$ on semi-Heyting algebras does not enjoy several nice properties of the implication on Heyting algebras or even on $BCK$-algebras. For example, the order on a semi-Heyting algebra is not determined by the operation of implication. Some of the properties of $\to$ in $\mathcal{SH}$ are contained in the next lemma.

**Lemma 1.1.** [4] *Let* $\mathbf{L} \in \mathcal{SH}$ *and* $a, b \in \mathbf{L}$.

(a) *If* $a \to b = 1$ *then* $a \leq b$.

(b) *If* $a \leq b$ *then* $a \leq a \to b$.

(c) $a = b$ *if and only if* $a \to b = b \to a = 1$.

(d) $1 \to a = a$.

**Proof.** From $a \to b = 1$ and (SH3), we get $a \wedge 1 = a \wedge b$, that is $a = a \wedge b$, and we have (a). For (b), by (SH3) and since $a \leq b$ it follows that $a = a \wedge (a \to b) \leq a \to b$. Property (c) is clear. To prove (d), observe that $a = 1 \wedge a = 1 \wedge (1 \to a) = 1 \to a$. □

Since congruences in semi-Heyting algebras are determine by filters [4, Th. 5.4], the subdirectly irreducible algebras in $\mathcal{SH}$ can be characterized by the following result, which is essential for the rest of the paper.

**Theorem 1.2.** [4, Th. 7.5] *Let* $\mathbf{L} \in \mathcal{SH}$ *with* $|\mathbf{L}| \geq 2$. *Then the following are equivalent:*

(a) $\mathbf{L}$ *is subdirectly irreducible.*

(b) $\mathbf{L}$ *has a unique coatom.*

In particular, if $\mathbf{L}$ is a subdirectly irreducible semi-Heyting algebra, then 1 is join-irreducible.

A semi-Heyting algebra $\mathbf{L} = \langle L, \vee, \wedge, \to, 0, 1\rangle$ is said to be a *semi-Heyting algebra with a Stone implication* if it satisfies the identity $(0 \to 1)^* \vee (0 \to 1)^{**} \approx 1$.

We denote by $\mathcal{ISSH}$ the variety of semi-Heyting algebras with a Stone implication.

In [4, Definition 8.1] Sankappanavar introduced the following subvarieties of $\mathcal{SH}$ by providing defining identities within $\mathcal{SH}$ for each of them:

| Subvariety | Defining identity within $\mathcal{SH}$ |
|---|---|
| $\mathcal{FTT}$ (**F**alse implies **T**rue is **T**rue) | $0 \to 1 \approx 1$ |
| $\mathcal{FTD}$ (**F**alse implies **T**rue is **D**ense) | $(0 \to 1)^* \approx 0$ |
| $\mathcal{QH}$ (Quasi-Heyting algebras) | $y \leq x \to y$ |
| $\mathcal{H}$ (Heyting algebras) | $(x \wedge y) \to x \approx 1$ |
| $\mathcal{SH}^S$ (Stone semi-Heyting algebras) | $x^* \vee x^{**} \approx 1$ |
| $\mathcal{SH}^B$ (Boolean semi-Heyting algebras) | $x \vee x^* \approx 1$ |
| $\mathcal{FTF}$ (**F**alse implies **T**rue is **F**alse) | $0 \to 1 \approx 0$ |
| $\mathcal{PTP}$ (**P**ossible implies **T**rue is **P**ossible) | $x \to 1 \approx x$ |
| $com\mathcal{SH}$ (Commutative semi-Heyting algebras) | $x \to y \approx y \to x$ |

He also introduced the subvariety $\mathcal{SH}^C$ of $\mathcal{SH}$ generated by chains and the subvarieties $\mathcal{FTT} \cap \mathcal{SH}^C$, $\mathcal{QH} \cap \mathcal{SH}^C$, $\mathcal{FTF} \cap \mathcal{SH}^C$ and $com\mathcal{SH} \cap \mathcal{SH}^C$.

The objective of this work is to prove that these subvarieties are in fact subvarieties of $\mathcal{ISSH}$. We study the relationships between them within $\mathcal{ISSH}$ and we determine the sublattice of $\mathcal{SH}$ generated by the above subvarieties. We also introduce and study new subvarieties of $\mathcal{ISSH}$.

If $\mathbf{L}$ is a totally ordered semi-Heyting algebra we say that $\mathbf{L}$ is a semi-Heyting chain. The following results were proved in [1].

**Theorem 1.3.** *An equational basis for $\mathcal{SH}^C$ relative to $\mathcal{SH}$ is given by the identity*

$$((x \vee (x \to y)) \to (x \to y)) \vee (y \to (x \wedge y)) \approx 1.$$

**Corollary 1.4.** *Every subdirectly irreducible algebra of $\mathcal{SH}^C$ is a chain.*

Now we prove some simple properties of Stone semi-Heyting algebras.

**Theorem 1.5.** *If $\mathbf{L}$ is a subdirectly irreducible Stone semi-Heyting algebra, then $0$ is $\wedge$-irreducible.*

**Proof.** Suppose that there exist $a, b \in L$ such that $a \wedge b = 0$. Suppose that $a \neq 0$. Since $\mathbf{L}$ satisfies the Stone identity, $a^* \vee a^{**} = 1$, and since 1 is $\vee$-irreducible, $a^* = 1$ or $a^{**} = 1$. But $a^* \neq 1$, so $a^{**} = 1$. Then $a^* = 0$, and thus $0 = b \wedge a^* = b \wedge (a \to 0) \overset{(SH3)}{=} b \wedge [(b \wedge a) \to (b \wedge 0)] = b \wedge (0 \to 0) = b$. $\square$

**Corollary 1.6.** *If* $\mathbf{L}$ *is a finite subdirectly irreducible semi-Heyting algebra, the* $\mathbf{L}$ *is a Stone algebra if and only if* $\mathbf{L}$ *has a unique atom.*

**Corollary 1.7.** *If* $\mathbf{L}$ *is a subdirectly irreducible Stone semi-Heyting algebra and* $|L| \leq 5$*, then* $\mathbf{L}$ *is a chain.*

In [4], the author proves that $\mathcal{PTP}^C = com\mathcal{SH}^C$, where $\mathcal{PTP}^C$ denotes the subvariety $\mathcal{PTP} \cap \mathcal{SH}^C$, and he asks if it is true that $\mathcal{PTP} = comSH$ ([4, Problem 14.11]). Let us prove that in general $\mathcal{PTP} = com\mathcal{SH}$.

**Theorem 1.8.** *Let* $\mathbf{L} \in \mathcal{SH}$*. The following conditions are equivalent:*

(1) $\mathbf{L} \models x \to y \approx y \to x$.

(2) $\mathbf{L} \models x \to 1 \approx x$.

(3) $\mathbf{L} \models y \wedge (x \to y) \approx x \wedge y$.

**Proof.** (1) $\Rightarrow$ (2) If $a \in L$, $a \to 1 = 1 \to a = a$.
(2) $\Rightarrow$ (3) Let $a, b \in L$. Then $b \wedge (a \to b) = b \wedge [(b \wedge a) \to (b \wedge b)] = b \wedge [(b \wedge a) \to b] = = b \wedge [(b \wedge a) \to (b \wedge 1)] = b \wedge (a \to 1) = b \wedge a$.
(3) $\Rightarrow$ (1) Let $a, b \in L$. Then $(a \to b) \wedge (b \to a) = (a \to b) \wedge [((a \to b) \wedge b) \to ((a \to b) \wedge a)] = = (a \to b) \wedge [(a \wedge b) \to (a \wedge b)] = (a \to b) \wedge 1 = (a \to b)$. Thus $a \to b \leq b \to a$. Similarly, $b \to a \leq a \to b$. So $a \to b = b \to a$. □

**Corollary 1.9.** $com\mathcal{SH} = \mathcal{PTP}$.

Once we have studied the variety in which $\to$ in commutative, it is natural to ask about the variety $asoc\mathcal{SH}$ in which $\to$ is associative. We will prove that in fact the identity $x \to (y \to z) \approx (x \to y) \to z$ characterizes the variety $\mathcal{V}(\overline{\mathbf{2}})$, where $\overline{\mathbf{2}}$ is the 2-element semi-Heyting chain that satisfies $0 \to 1 \approx 0$, and $\mathcal{V}(\overline{\mathbf{2}})$ is the variety generated by $\overline{\mathbf{2}}$.

**Lemma 1.10.** *If* $\mathbf{L} \in asoc\mathcal{SH}$*, then* $\mathbf{L}$ *satisfies* $x \to 1 \approx x$.

**Proof.** For $a \in L$, take $x = y = z = a$ in the identity $x \to (y \to z) \approx (x \to y) \to z$. □

**Corollary 1.11.** $asoc\mathcal{SH} \subseteq com\mathcal{SH}$.

**Theorem 1.12.** $asoc\mathcal{SH} = \mathcal{V}(\overline{\mathbf{2}})$

**Proof.** It is clear that $\overline{\mathbf{2}} \in asoc\mathcal{SH}$. So $\mathcal{V}(\overline{\mathbf{2}}) \subseteq asoc\mathcal{SH}$.

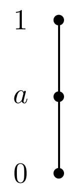
Let $\mathbf{L}$ be a subdirectly irreducible algebra in $asoc\mathcal{SH}$ with $|L| \geq 2$. Let $d \in L$ be the unique coatom in $L$ and let us prove that $d = 0$. Suppose that $d \neq 0$. We have that

$$0 \to (0 \to d) = (0 \to 0) \to d = 1 \to d = d.$$

From Corollary 1.11, $0 \to d = d \to 0 = d^* = 0$. So $d = 0 \to (0 \to d) = 0 \to 0 = 1$, a contradiction. Thus $|L| = 2$. By commutativity, we have that $0 \to 1 = 0$, so $\mathbf{L} \simeq \overline{\mathbf{2}}$. □

The following algebras will be used in section 2. It is routine to prove that they are subdirectly irreducible semi-Heyting algebras.

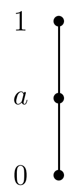
$\mathbf{L}_1$




| $\to$ | 0 | $a$ | 1 |
|---|---|---|---|
| 0 | 1 | 0 | 0 |
| $a$ | 0 | 1 | $a$ |
| 1 | 0 | $a$ | 1 |

In $\mathbf{L}_1$, $(0 \to 1)^* = 1$, so $\mathbf{L}_1 \in \mathcal{ISSH}$. On the other hand, it is clear that $\mathbf{L}_1 \notin \mathcal{FTD}$.

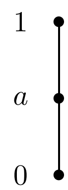
$\mathbf{L}_2$




| $\to$ | 0 | $a$ | 1 |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| $a$ | 0 | 1 | 1 |
| 1 | 0 | $a$ | 1 |

We have that $\mathbf{L}_2$ is a Heyting algebra, and $\mathbf{L}_1 \notin \mathcal{FTD}$.

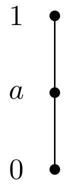
$\mathbf{L}_3$




| $\to$ | 0 | $a$ | 1 |
|---|---|---|---|
| 0 | 1 | $a$ | 1 |
| $a$ | 0 | 1 | 1 |
| 1 | 0 | $a$ | 1 |

It is clear that $\mathbf{L}_3$ satisfies $y \leq x \to y$, so $\mathbf{L}_3 \in \mathcal{QH}$. Since $a = 0 \to a \neq 1$, $\mathbf{L}_3 \notin \mathcal{H}$.

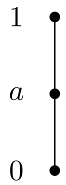
$\mathbf{L}_4$




| $\to$ | 0 | $a$ | 1 |
|---|---|---|---|
| 0 | 1 | $a$ | 1 |
| $a$ | 0 | 1 | $a$ |
| 1 | 0 | $a$ | 1 |

It is that $\mathbf{L}_4 \in \mathcal{FTT}$. Since $1 \neq a \to 1 = a$, $\mathbf{L}_4 \notin \mathcal{QH}$.

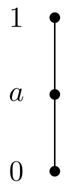
$\mathbf{L}_5$




| $\to$ | 0 | $a$ | 1 |
|---|---|---|---|
| 0 | 1 | $a$ | $a$ |
| $a$ | 0 | 1 | $a$ |
| 1 | 0 | $a$ | 1 |

We have that $\mathbf{L}_5 \in \mathcal{FTD}$ and $\mathbf{L}_5 \notin \mathcal{FTT}$.

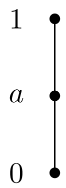
$\mathbf{L}_6$




| $\to$ | 0 | $a$ | 1 |
|---|---|---|---|
| 0 | 1 | 0 | 0 |
| $a$ | 0 | 1 | 1 |
| 1 | 0 | $a$ | 1 |

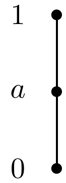
Observe that $a \to 1 \neq 1 \to a$, so $\mathbf{L}_6 \notin com\mathcal{SH}$.

$\mathbf{L}_7$




| $\to$ | 0 | $a$ | 1 |
|---|---|---|---|
| 0 | 1 | 1 | $a$ |
| $a$ | 0 | 1 | 1 |
| 1 | 0 | $a$ | 1 |

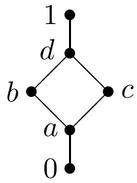
$\mathbf{L}_8$




| → | 0 | $a$ | 1 |
|---|---|---|---|
| 0 | 1 | $a$ | $a$ |
| $a$ | 0 | 1 | 1 |
| 1 | 0 | $a$ | 1 |

$\mathbf{L}_8 \in \mathcal{FTD}$.

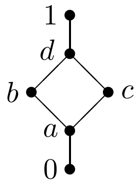
$\mathbf{L}_9$




| → | 0 | $a$ | $b$ | $c$ | $d$ | 1 |
|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| $a$ | 0 | 1 | 1 | 1 | 1 | 1 |
| $b$ | 0 | $c$ | 1 | $c$ | 1 | 1 |
| $c$ | 0 | $b$ | $b$ | 1 | 1 | 1 |
| $d$ | 0 | $a$ | $b$ | $c$ | 1 | 1 |
| 1 | 0 | $a$ | $b$ | $c$ | $d$ | 1 |

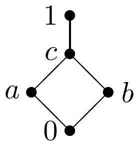
Observe that $\mathbf{L}_9$ is a Heyting algebra and it satisfies $x^* \vee x^{**} \approx 1$, so $\mathbf{L}_9 \in \mathcal{H}^S$.

$\mathbf{L}_{10}$




| → | 0 | $a$ | $b$ | $c$ | $d$ | 1 |
|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | 1 | $c$ | $b$ | $a$ | $a$ |
| $b$ | 0 | $c$ | 1 | $a$ | $b$ | $b$ |
| $c$ | 0 | $b$ | $a$ | 1 | $c$ | $c$ |
| $d$ | 0 | $a$ | $b$ | $c$ | 1 | $d$ |
| 1 | 0 | $a$ | $b$ | $c$ | $d$ | 1 |

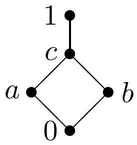
$\mathbf{L}_{11}$




| → | 0 | $a$ | $b$ | $c$ | 1 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 |
| $a$ | $b$ | 1 | $b$ | 1 | 1 |
| $b$ | $a$ | $a$ | 1 | 1 | 1 |
| $c$ | 0 | $a$ | $b$ | 1 | 1 |
| 1 | 0 | $a$ | $b$ | $c$ | 1 |

$\mathbf{L}_{11}$ is a Heyting algebra.

$\mathbf{L}_{12}$




| → | 0 | $a$ | $b$ | $c$ | 1 |
|---|---|---|---|---|---|
| 0 | 1 | $b$ | $a$ | 0 | 0 |
| $a$ | $b$ | 1 | 0 | $a$ | $a$ |
| $b$ | $a$ | 0 | 1 | $b$ | $b$ |
| $c$ | 0 | $a$ | $b$ | 1 | $c$ |
| 1 | 0 | $a$ | $b$ | $c$ | 1 |

## 2. Generating a sublattice of $\mathcal{ISSH}$

The objective of this section is to determine the sublattice generated by the subvarieties introduced in section 1 within the lattice of subvarieties of $\mathcal{SH}$.

**Lemma 2.1.** *Let* $\mathbf{L} \in \mathcal{SH}$.

(a) *If* $\mathbf{L} \models (x \wedge y) \to x \approx 1$ *then* $\mathbf{L} \models y \wedge (x \to y) \approx y$.

(b) *If* $\mathbf{L} \models y \wedge (x \to y) \approx y$ *then* $\mathbf{L} \models 0 \to 1 \approx 1$.

(c) *If* $\mathbf{L} \models 0 \to 1 \approx 1$ *then* $\mathbf{L} \models (0 \to 1)^* \approx 0$.

**Proof.** $y \wedge (x \to y) \overset{(SH3)}{=} y \wedge ((y \wedge x) \to (y \wedge y)) = y \wedge ((y \wedge x) \to y) = y \wedge 1 = y$, proving (a). (b) follows taking $x = 0$, $y = 1$. Finally, (c) is clear. □

**Lemma 2.2.** $\mathcal{H} \subsetneq \mathcal{QH} \subsetneq \mathcal{FTT} \subsetneq \mathcal{FTD} \subsetneq \mathcal{ISSH}$.

**Proof.** From Lemma 2.1, $\mathcal{H} \subseteq \mathcal{QH} \subseteq \mathcal{FTT} \subseteq \mathcal{FTD}$, and it is clear that $\mathcal{FTD} \subseteq \mathcal{ISSH}$. The algebras $\mathbf{L}_3$, $\mathbf{L}_4$ and $\mathbf{L}_5$ prove that $\mathcal{H} \neq \mathcal{QH}$, $\mathcal{QH} \neq \mathcal{FTT}$ and $\mathcal{FTT} \neq \mathcal{FTD}$. The algebra $\mathbf{L}_1 \in \mathcal{ISSH} \setminus \mathcal{FTD}$, so $\mathcal{FTD} \neq \mathcal{ISSH}$. □

**Lemma 2.3.** $com\mathcal{SH} \subsetneq \mathcal{FTF} \subsetneq \mathcal{ISSH}$.

**Proof.** Let $\mathbf{L} \in com\mathcal{SH}$. In $\mathbf{L}$, $0 \to 1 = 1 \to 0 = 0$, so $\mathbf{L} \in \mathcal{FTF}$. It is clear that $\mathcal{FTF} \subseteq \mathcal{ISSH}$ and consequently, $com\mathcal{SH} \subseteq \mathcal{FTF} \subseteq \mathcal{ISSH}$. Taking into account the algebras $\mathbf{L}_6$ and $\mathbf{L}_4$ we have that $com\mathcal{SH} \neq \mathcal{FTF}$ and $\mathcal{FTF} \neq \mathcal{ISSH}$. □

Let $\mathbf{2}$ be the 2-element semi-Heyting chain with universe $\{0, 1\}$ that satisfies $0 \to 1 \approx 1$, that is, $\mathbf{2}$ is the 2-element Boolean algebra, and let $\mathcal{T}$ denote the trivial variety. It is clear that $\mathcal{T} \subsetneq \mathcal{V}(\mathbf{2}) \subsetneq \mathcal{H}$.

Let us consider now the following identities.

$$[(x \vee x^*) \wedge (0 \to 1)] \vee [((x \to y) \leftrightarrow (y \to x)) \wedge (0 \to 1)^*] \approx 1 \qquad (E_1)$$

$$[(0 \to 1)^* \wedge (x \vee x^*)] \vee [((x \wedge y) \to y) \wedge (0 \to 1)] \approx 1 \qquad (E_2)$$

$$[((x \wedge y) \to y) \wedge (0 \to 1)] \vee [((x \to y) \leftrightarrow (y \to x)) \wedge (0 \to 1)^*] \approx 1 \quad (E_3)$$

$$[((x \wedge y) \to y) \wedge (0 \to 1)] \vee (0 \to 1)^* \approx 1 \quad (E_4)$$

$$(x \vee x^*) \vee (0 \to 1)^* \approx 1 \quad (E_5)$$

$$[(y \wedge (x \to y) \leftrightarrow y) \wedge (0 \to 1)] \vee [(x \vee x^*) \wedge (0 \to 1)^*] \approx 1 \quad (E_6)$$

$$[(y \wedge (x \to y) \leftrightarrow y) \wedge (0 \to 1)] \vee [((x \to y) \leftrightarrow (y \to x)) \wedge (0 \to 1)^*] \approx 1 \quad (E_7)$$

$$[(y \wedge (x \to y) \leftrightarrow y) \wedge (0 \to 1)] \vee (0 \to 1)^* \approx 1 \quad (E_8)$$

$$(0 \to 1) \vee [(0 \to 1)^* \wedge (x \vee x^*)] \approx 1 \quad (E_9)$$

$$(0 \to 1) \vee [(0 \to 1)^* \wedge ((x \to y) \leftrightarrow (y \to x))] \approx 1 \quad (E_{10})$$

$$(0 \to 1) \vee (0 \to 1)^* \approx 1 \quad (E_{11})$$

$$(0 \to 1)^{**} \vee [(0 \to 1)^* \wedge (x \vee x^*)] \approx 1 \quad (E_{12})$$

$$(0 \to 1)^{**} \vee [(0 \to 1)^* \wedge ((x \to y) \leftrightarrow (y \to x))] \approx 1 \quad (E_{13})$$

Let $\mathcal{E}_j$ denote the subvariety of $\mathcal{SH}$ defined by the identity $(E_j)$.

**Lemma 2.4.** $\mathcal{V}(\overline{\mathbf{2}}) \subsetneq \mathcal{SH}^B \subsetneq \mathcal{E}_2 \subsetneq \mathcal{E}_6 \subsetneq \mathcal{E}_9 \subsetneq \mathcal{E}_{12}$

**Proof.** Let $\mathbf{L} \in \mathcal{E}_2$ be subdirectly irreducible. For $a, b \in L$,

$$[(0 \to 1)^* \wedge (a \vee a^*)] \vee [((a \wedge b) \to b) \wedge (0 \to 1)] = 1.$$

Then $(0 \to 1)^* \wedge (a \vee a^*) = 1$ or $((a \wedge b) \to b) \wedge (0 \to 1) = 1$.

If $((a \wedge b) \to b) \wedge (0 \to 1) = 1$ then $(a \wedge b) \to b = 1$ and $0 \to 1 = 1$. So $b \wedge (a \to b) = b$ and $0 \to 1 = 1$. Thus $\mathbf{L} \in \mathcal{E}_6$, that is, $\mathcal{E}_2 \subseteq \mathcal{E}_6$.

The other inclusions are similar.

Let us see that $\mathcal{E}_2 \neq \mathcal{E}_6$. The algebra $\mathbf{L}_3$ satisfies the identities $y \wedge (x \to y) \approx y$ and $0 \to 1 \approx 1$. So $\mathbf{L}_3 \in \mathcal{E}_6$. But if we take $x = 0$ and $y = a$ in the identity $(E_2)$, we obtain $[(0 \to 1)^* \wedge (0 \vee 0^*)] \vee [(0 \to a) \wedge (0 \to 1)] = 0 \vee [a \wedge 1] = a \neq 1$. Thus $\mathbf{L}_3 \notin \mathcal{E}_2$.

For the rest of the inequalities, it is enough to consider the algebras $\mathbf{2}$, $\mathbf{L}_2$, $\mathbf{L}_4$ and $\mathbf{L}_8$. □

**Lemma 2.5.** $com\mathcal{SH} \subsetneq \mathcal{E}_1 \subsetneq \mathcal{E}_3 \subsetneq \mathcal{E}_7 \subsetneq \mathcal{E}_{10} \subsetneq \mathcal{E}_{13}$.

**Proof.** Let us prove that $\mathcal{E}_1 \subsetneq \mathcal{E}_3$. Let $\mathbf{L} \in \mathcal{E}_1$ be subdirectly irreducible and $a, b \in L$. If $(a \vee a^*) \wedge (0 \to 1) = 1$ then $a \vee a^* = 0 \to 1 = 1$. Then $a = 1$ or $a = 0$. In both cases, $(a \wedge b) \to b = 0 \to 1 = 1$. So $\mathbf{L} \in \mathcal{E}_3$.

The algebra $\mathbf{L}_2$ belongs to $\mathcal{E}_3$, but if we take $x = y = a$ in $(E_1)$, we obtain $(a \vee a^*) \wedge (0 \to 1) = a \neq 1$, so $\mathbf{L}_2 \notin \mathcal{E}_1$. Consequently $\mathcal{E}_1 \subsetneq \mathcal{E}_3$.

The other cases are similar and the corresponding inequalities follow considering the algebras $\mathbf{L}_3$, $\mathbf{L}_4$, $\mathbf{L}_5$ and the algebra $\mathbf{2}$. □

**Lemma 2.6.** $\mathcal{FTF} \subsetneq \mathcal{E}_5 \subsetneq \mathcal{E}_4 \subsetneq \mathcal{E}_8 \subsetneq \mathcal{E}_{11} \subsetneq \mathcal{ISSH}$.

**Proof.** We only prove that $\mathcal{E}_5 \subsetneq \mathcal{E}_4$. Let $\mathbf{L} \in \mathcal{E}_5$ be subdirectly irreducible and let $a, b \in L$. We have that $\mathbf{L}$ satisfies $(x \vee x^*) \vee (0 \to 1)^* \approx 1$. If $0 \to 1 = 0$ we are done. If $0 \to 1 = 1$ then $a \vee a^* = 1$ and as in the previous proof, $(a \wedge b) \to b = 1$. Finally, the case $0 \to 1 = a$ with $a \notin \{0, 1\}$ is not possible, since otherwise we would have $(a \vee a^*) \vee (0 \to 1)^* = a \vee a^* \neq 1$. Therefore, $\mathbf{L} \in \mathcal{E}_4$.

The algebra $\mathbf{L}_2$ belongs to $\mathcal{E}_4$, but if we take $x = a$ in $(E_5)$, we see that $\mathbf{L}_2 \notin \mathcal{E}_5$. Hence, $\mathcal{E}_5 \subsetneq \mathcal{E}_4$.

The other relations can be checked taking into account Lemma 2.1 and by using the algebras $\mathbf{2}$, $\mathbf{L}_3$, $\mathbf{L}_4$ and $\mathbf{L}_5$. □

In a similar way the following relations can be proved.

**Lemma 2.7.**

(1) $\mathcal{V}(\overline{\mathbf{2}}) \subsetneq com\mathcal{SH}$

(2) $\mathcal{V}(\mathbf{2}) \subsetneq \mathcal{SH}^B \subsetneq \mathcal{E}_1 \subsetneq \mathcal{E}_5$

(3) $\mathcal{H} \subsetneq \mathcal{E}_2 \subsetneq \mathcal{E}_3 \subsetneq \mathcal{E}_4$

(4) $\mathcal{QH} \subsetneq \mathcal{E}_6 \subsetneq \mathcal{E}_7 \subsetneq \mathcal{E}_8$

(5) $\mathcal{FTT} \subsetneq \mathcal{E}_9 \subsetneq \mathcal{E}_{10} \subsetneq \mathcal{E}_{11}$

(6) $\mathcal{FTD} \subsetneq \mathcal{E}_{12} \subsetneq \mathcal{E}_{13} \subsetneq \mathcal{ISSH}$

For a given variety $\mathcal{V}$, let $\mathcal{V}^C$ denote the variety $\mathcal{V} \cap \mathcal{SH}^C$, and similarly, let $\mathcal{V}^S$ denote the variety $\mathcal{V} \cap \mathcal{SH}^S$.

**Theorem 2.8.** $\mathcal{SH}^C \subsetneq \mathcal{SH}^S \subsetneq \mathcal{ISSH}$

**Proof.** By Corollary 1.4, $\mathcal{SH}^C \subseteq \mathcal{SH}^S$, and it is clear that $\mathcal{SH}^S \subseteq \mathcal{ISSH}$.

The algebra $\mathbf{L}_9 \in \mathcal{SH}^S$. Since $\mathbf{L}_9$ is a subdirectly irreducible algebra which is not a chain, $\mathbf{L}_9 \notin \mathcal{SH}^C$ (Corollary 1.4).

Similarly, the algebra $\mathbf{L}_{11} \in \mathcal{ISSH}$, but $\mathbf{L}_{11} \notin \mathcal{SH}^S$, since it does not have a unique atom. □

**Corollary 2.9.**

(a) $\mathcal{H}^C \subsetneqq \mathcal{H}^S \subsetneqq \mathcal{H}$

(b) $\mathcal{QH}^C \subsetneqq \mathcal{QH}^S \subsetneqq \mathcal{QH}$

(c) $\mathcal{FTT}^C \subsetneqq \mathcal{FTT}^S \subsetneqq \mathcal{FTT}$

(d) $\mathcal{FTD}^C \subsetneqq \mathcal{FTD}^S \subsetneqq \mathcal{FTD}$

**Corollary 2.10.**

(a) $com\mathcal{SH}^C \subsetneqq com\mathcal{SH}^S \subsetneqq com\mathcal{SH}$

(b) $\mathcal{FTF}^C \subsetneqq \mathcal{FTF}^S \subsetneqq \mathcal{FTF}$

**Proof.** We shall prove item (a). By Theorem 2.8, $com\mathcal{SH}^C \subseteq com\mathcal{SH}^S \subseteq com\mathcal{SH}$. Now, the algebra $\mathbf{L}_{10} \in com\mathcal{SH}^S$. But, by Theorem 1.4, $\mathbf{L}_{10} \notin com\mathcal{SH}^C$. On the other hand, the algebra $\mathbf{L}_{12} \in com\mathcal{SH}$, while $\mathbf{L}_{12} \notin com\mathcal{SH}^S$ since it has no a unique atom. □

**Corollary 2.11.** $\mathcal{E}_j^C \subsetneqq \mathcal{E}_j^S \subsetneqq \mathcal{E}_j, 1 \leq j \leq 13$

**Proof.** We prove only the case $j = 1$. By Theorem 2.8, $\mathcal{E}_1^C \subseteq \mathcal{E}_1^S \subseteq \mathcal{E}_1$. The algebra $\mathbf{L}_{10}$ is commutative, so in particular, $\mathbf{L}_{10} \in \mathcal{E}_1^S$. Since $\mathcal{E}_1^C \subseteq \mathcal{SH}^C$, by Theorem 2.8, $\mathbf{L}_{10} \notin \mathcal{E}_1^C$. So $\mathcal{E}_1^C \subsetneqq \mathcal{E}_1^S$. On the other hand, the algebra $\mathbf{L}_{12}$ is commutative, an then $\mathbf{L}_{12} \in \mathcal{E}_1$, but $a^* \vee a^{**} \neq 1$, so $\mathbf{L}_{12} \notin \mathcal{SH}^S$. □

The following lemma will be used in the rest of the section.

**Lemma 2.12.** *Let* $\mathbf{L} \in \mathcal{SH}^C$ *be subdirectly irreducible. If* $0 \to 1 = c$ *with* $c \in L \setminus \{0, 1\}$ *then* $\mathbf{L}$ *does not satisfy any of the identities* ($E_1$) *to* ($E_{11}$).

**Proof.** If $\mathbf{L} \in \mathcal{SH}^C$ is subdirectly irreducible, $\mathbf{L}$ is a chain. Since $0 \to 1 = c$ with $c \in L - \{0, 1\}$, $(0 \to 1)^* = 0$. The result follows if we take $x = y = c$ in any of he identities $(E_1)$ to $(E_{11})$. □

In what follows we will find the join and the meet in the lattice of subvarieties of $\mathcal{ISSH}$ of each pair of subvarieties previously defined. Observe that an equational basis for $V(\mathbf{2})$, modulo $\mathcal{SH}$, is given by $x \vee x^* \approx 1$ and $0 \to 1 \approx 1$ ([4, Corollary 9.3]), and an equational base for $V(\overline{\mathbf{2}})$, modulo $\mathcal{SH}$, is given by $x \vee x^* \approx 1$ and $0 \to 1 \approx 0$ ([4, Corollary 9.4]). Thus $V(\mathbf{2}) = \mathcal{SH}^B \cap \mathcal{FTT}$ and $V(\overline{\mathbf{2}}) = \mathcal{SH}^B \cap \mathcal{FTF}$

In [4] it is shown the following result, where $\mathcal{V}(A, B)$ (respectively $\mathcal{V}(\mathcal{A}, \mathcal{B})$) denotes the variety generated by the algebras $A$ and $B$ (respectively by the subvarieties $\mathcal{A}$ and $\mathcal{B}$).

**Lemma 2.13.**

(a) $\mathcal{V}(\mathbf{2}, \overline{\mathbf{2}}) = \mathcal{SH}^B$.

(b) $\mathcal{V}(\mathbf{2}) \cap \mathcal{V}(\overline{\mathbf{2}}) = \mathcal{T}$.

**Lemma 2.14.**

*(a)* $\mathcal{H} \cap \mathcal{SH}^B = \mathcal{V}(\mathbf{2})$

*(b)* $\mathcal{V}(\mathcal{H}, \mathcal{SH}^B) = \mathcal{E}_2$

**Proof.** It is clear that $\mathbf{2} \in \mathcal{H} \cap \mathcal{SH}^B$. Let $\mathbf{L} \in \mathcal{H} \cap \mathcal{SH}^B$ be subdirectly irreducible. By Lemma 2.13, $\mathbf{L} \simeq \mathbf{2}$ or $\mathbf{L} \simeq \overline{\mathbf{2}}$. Since $\mathbf{L} \in \mathcal{H}$, $\mathbf{L} \simeq \mathbf{2}$. So we have (a).

In order to prove (b), let $\mathbf{L} \in \mathcal{E}_2$ be subdirectly irreducible. Suppose that $0 \to 1 = 0$. Then for $x \in L$, we obtain in $(E_2)$, $x \vee x^* \approx 1$. So $\mathbf{L} \in \mathcal{SH}^B$. If $0 \to 1 = 1$, then for $x, y \in L$ we obtain in $(E_2)$, $(x \wedge y) \to y \approx 1$, and consequently, $\mathbf{L} \in \mathcal{H}$. In addition, from Lemmas 2.4 and 2.7, $\mathcal{SH}^B \subseteq \mathcal{E}_2$ and $\mathcal{H} \subseteq \mathcal{E}_2$. □

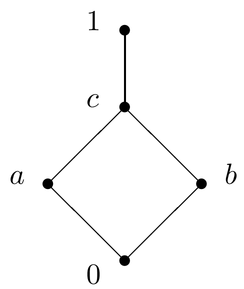
In a similar way, by using Lemma 2.12 and the previous results and examples, it can be proved that:

**Lemma 2.15.**

1. (a) $\mathcal{SH}^B \cap com\mathcal{SH} = \mathcal{V}(\overline{\mathbf{2}})$
   (b) $\mathcal{V}(\mathcal{SH}^B, com\mathcal{SH}) = \mathcal{E}_1$
2. (a) $\mathcal{QH} \cap \mathcal{E}_2 = \mathcal{H}$
   (b) $\mathcal{V}(\mathcal{QH}, \mathcal{E}_2) = \mathcal{E}_6$
3. (a) $\mathcal{E}_2 \cap \mathcal{E}_1 = \mathcal{BSH}$
   (b) $\mathcal{V}(\mathcal{E}_2, \mathcal{E}_1) = \mathcal{E}_3$
4. (a) $\mathcal{E}_1 \cap \mathcal{FTF} = com\mathcal{SH}$
   (b) $\mathcal{V}(\mathcal{E}_1, \mathcal{FTF}) = \mathcal{E}_5$
5. (a) $\mathcal{E}_6 \cap \mathcal{FTT} = \mathcal{QH}$
   (b) $\mathcal{V}(\mathcal{E}_6, \mathcal{FTT}) = \mathcal{E}_9$
6. (a) $\mathcal{E}_6 \cap \mathcal{E}_3 = \mathcal{E}_2$
   (b) $\mathcal{V}(\mathcal{E}_6, \mathcal{E}_3) = \mathcal{E}_7$
7. (a) $\mathcal{E}_3 \cap \mathcal{E}_5 = \mathcal{E}_1$
   (b) $\mathcal{V}(\mathcal{E}_3, \mathcal{E}_5) = \mathcal{E}_4$
8. (a) $\mathcal{FTD} \cap \mathcal{E}_9 = \mathcal{FTT}$
   (b) $\mathcal{V}(\mathcal{FTD}, \mathcal{E}9) = \mathcal{E}_{12}$
9. (a) $\mathcal{E}_9 \cap \mathcal{E}_7 = \mathcal{E}_6$
   (b) $\mathcal{V}(\mathcal{E}_9, \mathcal{E}_7) = \mathcal{E}_{10}$
10. (a) $\mathcal{E}_7 \cap \mathcal{E}_4 = \mathcal{E}_3$
    (b) $\mathcal{V}(\mathcal{E}_7, \mathcal{E}_4) = \mathcal{E}_8$
11. (a) $\mathcal{E}_{12} \cap \mathcal{E}_{10} = \mathcal{E}_9$
    (b) $\mathcal{V}(\mathcal{E}_{12}, \mathcal{E}_{10}) = \mathcal{E}_{13}$
12. (a) $\mathcal{E}_{10} \cap \mathcal{E}_8 = \mathcal{E}_7$
    (b) $\mathcal{V}(\mathcal{E}_{10}, \mathcal{E}_8) = \mathcal{E}_{11}$
13. (a) $\mathcal{E}_{13} \cap \mathcal{E}_{11} = \mathcal{E}_{10}$
    (b) $\mathcal{V}(\mathcal{E}_{13}, \mathcal{E}_{11}) = \mathcal{ISSH}$

Observe that $\mathcal{ISSH} \subsetneqq \mathcal{SH}$, as the following example shows.

**L**

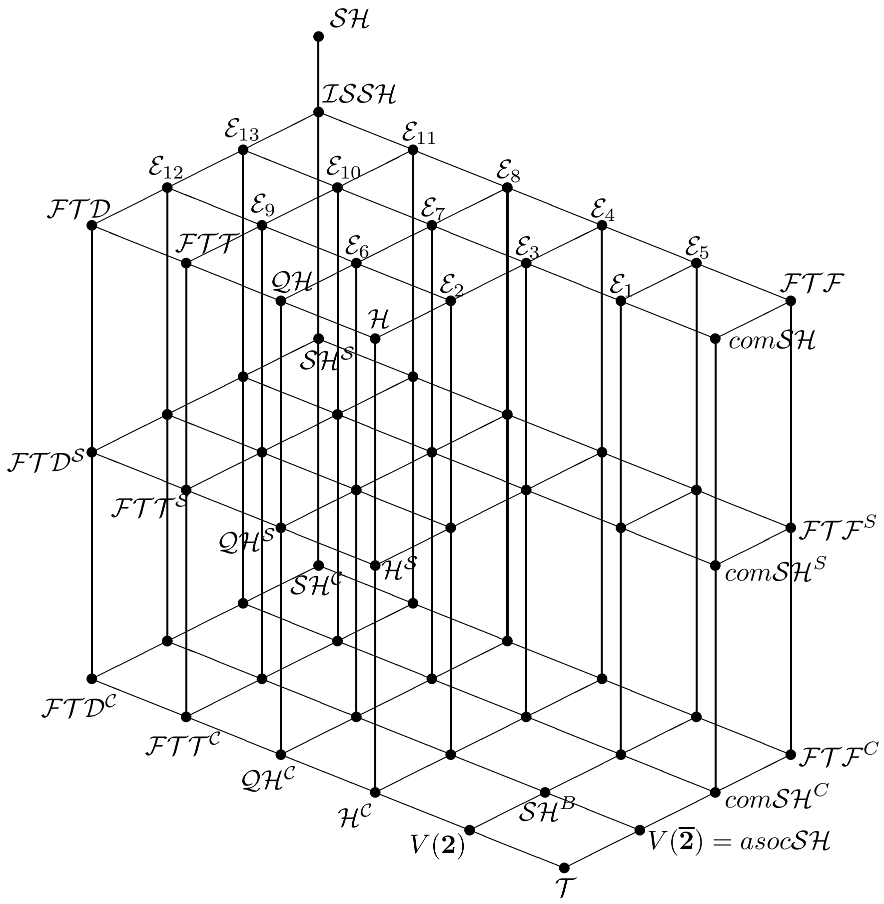



| $\to$ | 0 | $a$ | $b$ | $c$ | 1 |
|---|---|---|---|---|---|
| 0 | 1 | $b$ | $a$ | $a$ | $a$ |
| $a$ | $b$ | 1 | 0 | $a$ | $a$ |
| $b$ | $a$ | $a$ | 1 | 1 | 1 |
| $c$ | 0 | $a$ | $b$ | 1 | 1 |
| 1 | 0 | $a$ | $b$ | $c$ | 1 |

We have that **L** is a semi-Heyting algebra, but $(0 \to 1)^{**} \vee (0 \to 1)^* = a^{**} \vee a^* = b^* \vee b = a \vee b = c \neq 1$, so $\mathbf{L} \notin \mathcal{ISSH}$.

Thus we have the following theorem.

**Theorem 2.16.** *The order relation between the subvarieties previously defined is the one depicted in the following figure.*




## References


[1] M. Abad, J.M. Cornejo and J.P. Diaz Varela, *The Variety Generated by Semi-Heyting Chains*, submitted for publication.

[2] R. Balbes and P.H. Dwinger, Distributive lattices, University of Missouri Press, Columbia, 1974.

[3] S. Burris and H.P. Sankappanavar, A Course in Universal Algebra, Graduate Texts in Mathematics **78**, Springer-Verlag, New York, 1981.

[4] H.P. Sankappanavar, *Semi-Heyting Algebras: An Abstraction From Heyting Algebras.* Actas del IX Congreso A. Monteiro, Bahía Blanca, 2007, pp. 33–66.

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