VECTOR BUNDLES ON REAL ALGEBRAIC CURVES

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Abstract. We prove that any topological real line bundle on a compact real algebraic curve $X$ is isomorphic to an algebraic line bundle. The result is then generalized to vector bundles of an arbitrary constant rank. As a consequence we prove that any continuous map from $X$ into a real Grassmannian can be approximated by regular maps.

1. Introduction. Throughout this paper $X$ denotes a compact real algebraic curve, that is, a compact 1-dimensional algebraic subset of $\mathbb{R}^d$ for some $d \in \mathbb{N}$. We refer to [1] for terminology and background material on real algebraic geometry. In this paper all vector bundles are real vector bundles. Recall that algebraic vector bundles on $X$ correspond to finitely generated projective modules over the ring of real-valued regular functions on $X$, cf. [1], p. 302. Our main goal is the following:

Theorem 1.1. Any topological line bundle on $X$ is isomorphic to an algebraic line bundle.

Theorem 1.1 is proved in section 2. It can be easily generalized.

Corollary 1.2. Any topological constant rank vector bundle on $X$ is isomorphic to an algebraic vector bundle.

Proof. Any topological vector bundle on $X$ of constant rank $r \geq 1$ splits off a trivial vector bundle of rank $r - 1$, since $\dim(X) = 1$. Hence it suffices to apply Theorem 1.1.

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As a consequence of Corollary 1.2, we obtain a counterpart of the classical Weierstrass approximation theorem for maps from $X$ into the Grassmann variety $G_{n,k}$ of $k$-dimensional vector subspaces of $\mathbb{R}^n$.

**Corollary 1.3.** Let $f : X \to G_{n,k}$ be a continuous map. Each neighborhood of $f$ in the compact-open topology contains a regular map.

**Proof.** It suffices to show that the pullback vector bundle $f^*\gamma_{n,k}$ on $X$, where $\gamma_{n,k}$ is the tautological vector bundle on $G_{n,k}$, is isomorphic to an algebraic vector bundle, cf. [1, Theorem 13.3.1]. This however follows from Corollary 1.2.

Since the real variety $G_{2,1}$ is biregularly isomorphic to the unit circle $S^1 = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \}$, we immediately get:

**Corollary 1.4.** Let $f : X \to S^1$ be a continuous map. Each neighborhood of $f$ in the compact-open topology contains a regular map.

All the results above are proved in [1] under the assumption that the curve $X$ is nonsingular. The arguments presented in [1] do not directly generalize to yield Theorem 1.1.

**Corollary 1.5.** For every cohomology class $u$ in $H^1(X; \mathbb{Z}/2)$, there exists a regular map $f : X \to S^1$ such that $f^*(s_1) = u$, where $s_1$ is the unique generator of the cohomology group $H^1(S^1; \mathbb{Z}/2) \cong \mathbb{Z}/2$.

**Proof.** There is a one-to-one correspondence between the homotopy classes of continuous maps from $X$ into $S^1$ and the cohomology classes in $H^1(X; \mathbb{Z})$, cf., [2, p. 300]. Since the reduction modulo 2 homomorphism $H^1(X; \mathbb{Z}) \to H^1(X; \mathbb{Z}/2)$ is surjective, it follows that each cohomology class in $H^1(X; \mathbb{Z}/2)$ is of the form $f^*(s_1)$ for some continuous map $f : X \to S^1$. According to Corollary 1.4, the map $f$ can be assumed to be regular.

Let us note that Corollary 1.5 implies Theorem 1.1. Indeed, let $\xi$ be a topological line bundle on $X$. The first Stiefel–Whitney class $w_1(\gamma_{2,1})$ of the tautological line bundle $\gamma_{2,1}$ on $G_{2,1}$ generates the cohomology group $H^1(G_{2,1}; \mathbb{Z}/2)$. According to Corollary 1.5, there exists a regular map $f : X \to G_{2,1}$ satisfying $w_1(\xi) = f^*(w_1(\gamma_{2,1})) = w_1(f^*\gamma_{2,1})$. Since topological line bundles are classified by the first Stiefel-Whitney class (cf. [3, Proposition 3.10]), it follows that $\xi$ is isomorphic to the algebraic line bundle $f^*\gamma_{2,1}$. However, we do not know how to prove Corollary 1.5 without making use of Theorem 1.1.
2. Line bundles on real algebraic curves. We first recall a useful construction of algebraic line bundles on an arbitrary affine real algebraic variety V. Lemma 2.1 below is a special case of [1] Theorem 12.1.11.

**Lemma 2.1.** Let \( \{ U_1, \ldots, U_r \} \) be a Zariski open cover of V and let \( h_{ij} : U_j \rightarrow \mathbb{R} \) be a regular function satisfying \( h_{ij}(U_i \cap U_j) \subset \mathbb{R} \setminus \{0\} \) for \( 1 \leq i,j \leq r \). Assume that \( h_{ij} \cdot h_{jk} = h_{ik} \) on \( U_j \cap U_k \) for all \( i,j,k \), and \( h_{ii}(x) = 1 \) for all \( i \) and \( x \) in \( U_i \). Let

\[
E = \{(x, (v_1, \ldots, v_r)) \in V \times \mathbb{R}^r : v_i = h_{ij}(x)v_j \text{ for } x \in U_j, 1 \leq i, j \leq r\}
\]

and let \( p : E \rightarrow V \) be defined by \( p(x, (v_1, \ldots, v_r)) = x \). Then \( \xi = (E, p, V) \) is an algebraic line subbundle of the product vector bundle on V with total space \( V \times \mathbb{R}^r \), and the map

\[
U_i \times \mathbb{R} \rightarrow p^{-1}(U_i), (x, v) \mapsto (h_{i1}(x)v, \ldots, h_{ri}(x)v)
\]

is an algebraic trivialization of \( \xi \) over \( U_i \) for \( 1 \leq i \leq r \).

For any vector bundle \( \eta \) and any global section \( s \) of \( \eta \), let \( Z(s) \) denote the zero locus of \( s \).

The set \( \text{Reg}(X) \) of nonsingular points of \( X \) in dimension 1 is a Zariski open subset of \( X \), cf. [1] p. 69]. Furthermore, \( \text{Reg}(X) \) is a 1-dimensional \( C^\infty \) manifold.

**Lemma 2.2.** Let \( x_0 \) be a point in \( \text{Reg}(X) \). There exists an algebraic line bundle \( \xi = (E, p, X) \) on \( X \) which admits an algebraic section \( s : X \rightarrow E \) such that \( Z(s) = \{x_0\} \) and the restriction of \( s \) to \( \text{Reg}(X) \) is transverse to the zero section of \( \xi \).

**Proof.** Let \( \mathcal{R}_X \) be the sheaf of real-valued regular functions on \( X \). For any point \( x \) on \( X \), we identify the stalk \( \mathcal{R}_{X,x} \) with the localization of the ring \( \mathcal{R}_X(X) \) at the maximal ideal

\[
m_x = \{ f \in \mathcal{R}_X(X) : f(x) = 0 \},
\]

cf. [1] Proposition 3.2.3]. Since the point \( x_0 \) is in \( \text{Reg}(X) \), the stalk \( \mathcal{R}_{X,x_0} \) is a regular local ring of dimension 1 and thus a principal ideal domain. In particular, the ideal \( m_{x_0} \mathcal{R}_{X,x_0} \) of the ring \( \mathcal{R}_{X,x_0} \) is principal. Thus we can find a regular function \( f_1 \) in \( m_{x_0} \) and a Zariski open neighborhood \( U_1 \) of \( x_0 \) in \( \text{Reg}(X) \) such that

\[
m_{x_0} \mathcal{R}_X(U_1) = (f_1) \mathcal{R}_X(U_1).
\]

In particular, \( f_1|_{U_1} : U_1 \rightarrow \mathbb{R} \) is a \( C^\infty \) function for which 0 in \( \mathbb{R} \) is a regular value and \( (f_1|_{U_1})^{-1}(0) = \{x_0\} \).

Let \( f_2 \) be any regular function in \( m_{x_0} \) with \( f_2^{-1}(0) = \{x_0\} \), e.g., a polynomial given by the formula \( \|x - x_0\|^2 \), where \( \| \cdot \| \) denotes the euclidean metric.
in \( \mathbb{R}^d \). We have
\[
f_2|_{U_1} = h_{21}f_1|_{U_1}
\]
for some regular function \( h_{21} : U_1 \rightarrow \mathbb{R} \). If \( U_2 = X \setminus \{x_0\} \), then
\[
h_{12} = \frac{f_1}{f_2} : U_2 \rightarrow \mathbb{R}
\]
is a regular function on \( U_2 \). By construction, the sets \( h_{21}(U_1 \cap U_2) \) and \( h_{12}(U_1 \cap U_2) \) are contained in \( \mathbb{R} \setminus \{0\} \). Define \( h_{11} : U_1 \rightarrow \mathbb{R} \) and \( h_{22} : U_2 \rightarrow \mathbb{R} \) to be constant functions identically equal to 1. Let \( \xi = (E, p, X) \) be the algebraic line bundle on \( X \) determined, as in Lemma 2.1, by the Zariski open cover \( \{U_1, U_2\} \) of \( X \) and the regular functions \( h_{ij} \). Note that
\[
s : X \rightarrow E, \ s(x) = (x, (f_1(x), f_2(x)))
\]
is an algebraic section of \( \xi \) with \( Z(s) = \{x_0\} \). On the set \( U_1 \), the section \( s \) is represented by the map
\[
U_1 \rightarrow U_1 \times \mathbb{R}, \ x \mapsto (x, f_1(x)),
\]
and hence the restriction of \( s \) to \( \text{Reg}(X) \) is transverse to the zero section of \( \xi \). \( \square \)

We will now give a convenient description of the first cohomology group \( H^1(X; \mathbb{Z}/2) \) of the curve \( X \). The subset \( X \setminus \text{Reg}(X) \) of \( X \) is finite. If \( X \) has nonsingular connected components, we choose one arbitrary point in each of those and denote the set of such points by \( Z \). The curve \( X \) can be regarded as a graph (1-dimensional CW complex) with \( (X \setminus \text{Reg}(X)) \cup Z \) as the set of vertices. This assertion is a straightforward consequence of the triangulation theorem for semi-algebraic sets, cf. [1, Theorem 9.2.1].

**Lemma 2.3.** There exist subgraphs \( X_1, \ldots, X_n \) of \( X \) such that each \( X_i \) is homeomorphic to the unit circle \( S^1 \), and the inclusion maps \( X_i \hookrightarrow X \) induce an isomorphism
\[
\varphi : H^1(X; \mathbb{Z}/2) \rightarrow \bigoplus_{i=1}^n H^1(X_i; \mathbb{Z}/2)
\]

**Proof.** Let \( K \) be a connected 1-dimensional component of \( X \) and let \( T \) be a maximal tree of the graph \( K \). The quotient map \( q : K \rightarrow K/T \) is a homotopy equivalence and the quotient space \( K/T \) is homeomorphic to the wedge sum of a finite number of pointed circles, [2, p. 153]. Each such pointed circle corresponds to a subset of \( K/T \) of the form \( q(C) \), where \( C \) is a subgraph of \( K \) homeomorphic to the unit circle. The inclusion maps \( q(C) \hookrightarrow K/T \) induce an isomorphism
\[
\psi : H^1(K/T; \mathbb{Z}/2) \rightarrow \bigoplus_C H^1(q(C); \mathbb{Z}/2)
\]
If \( q_C : C \rightarrow q(C) \) is the restriction of the map \( q \), then the homomorphism
\[
\alpha = \bigoplus_C q_C^* : \bigoplus_C H^1(q(C); \mathbb{Z}/2) \rightarrow \bigoplus_C H^1(C; \mathbb{Z}/2)
\]
is an isomorphism. The homomorphism
\[
q^* : H^1(K/T; \mathbb{Z}/2) \rightarrow H^1(K; \mathbb{Z}/2)
\]
is an isomorphism, the quotient map being a homotopy equivalence. Finally, the inclusion maps \( C \hookrightarrow K \) induce a homomorphism
\[
\varphi_K : H^1(K; \mathbb{Z}/2) \rightarrow \bigoplus_C H^1(C; \mathbb{Z}/2)
\]
satisfying \( \varphi_K \circ q^* = \alpha \circ \psi \). Consequently, \( \varphi_K \) is an isomorphism.

The assertion of the lemma follows, because \( X \) has finitely many connected components.

Proof of Theorem 1.1. The isomorphism classes of topological line bundles on \( X \) form a group, denoted \( \text{Vect}^1(X) \), with tensor product as the group operation. The first Stiefel–Whitney class gives a group isomorphism between \( \text{Vect}^1(X) \) and the first cohomology group \( H^1(X; \mathbb{Z}/2) \), cf. [3, Proposition 3.10]. Also, note that the isomorphism classes of algebraic vector bundles form a subgroup of \( \text{Vect}^1(X) \). Hence, in view of Lemma 2.3, it remains to construct for each \( i = 1, \ldots, n \) an algebraic line bundle \( \xi_i \) on \( X \) with \( w_1(\xi_i|_{X_i}) \neq 0 \) and \( w_1(\xi_i|_{X_j}) = 0 \) for all \( j \neq i \) (note that \( H^1(X_i; \mathbb{Z}/2) \cong \mathbb{Z}/2 \)). Such a line bundle \( \xi_i \) can be obtained as follows.

Let \( x_i \) be a point in
\[
(X_i \cap \text{Reg}(X)) \setminus \bigcup_{j \neq i} X_j
\]
and let \( \xi = (E, p, X) \) be an algebraic line bundle on \( X \) as in Lemma 2.2 with \( x_0 = x_i \). There exists an algebraic section \( s : X \rightarrow E \) such that \( Z(s) = \{x_i\} \) and the restriction of \( s \) to \( \text{Reg}(X) \) is transverse to the zero section of \( \xi \). It follows that the line bundle \( \xi|_{X_j} \) is trivial and \( w_1(\xi|_{X_j}) = 0 \) for \( j \neq i \).

Suppose for a moment that the line bundle \( \xi|_{X_i} \) is trivial, and let \( \theta : p^{-1}(X_i) \rightarrow X_i \times \mathbb{R} \) be a topological trivialization of \( \xi|_{X_i} \). Then \( \theta(s(x)) = (x, f(x)) \) for each \( x \) in \( X_i \), where \( f : X_i \rightarrow \mathbb{R} \) is a continuous function. By construction, \( f^{-1}(0) = \{x_i\} \). The function \( f \) does not change sign on \( X_i \setminus \{x_i\} \), the set \( X_i \setminus \{x_i\} \) being homeomorphic to \( \mathbb{R} \). This however is impossible since \( s \) is transverse to the zero section of \( \xi \) in a neighborhood of \( x_i \). Consequently, the line bundle \( \xi|_{X_i} \) is nontrivial and \( w_1(\xi|_{X_i}) \neq 0 \).

We complete the proof by setting \( \xi_i = \xi \).
References


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