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BOREL SETS WITHOUT PERFECTLY MANY OVERLAPPING TRANSLATIONS

Abstract. We study the existence of Borel sets $B \subseteq \omega^2$ admitting a sequence $\langle \eta_\alpha : \alpha < \lambda \rangle$ of distinct elements of $\omega^2$ such that $| (\eta_\alpha + B) \cap (\eta_\beta + B) | \geq 6$ for all $\alpha, \beta < \lambda$ but with no perfect set of such $\eta$’s. Our result implies that under the Martin Axiom, if $\aleph_\alpha < \mathfrak{c}$, $\alpha < \omega_1$ and $3 \leq \iota < \omega$, then there exists a $\Sigma^0_2$ set $B \subseteq \omega^2$ which has $\aleph_\alpha$ many pairwise $2\iota$–nondisjoint translations but not a perfect set of such translations. Our arguments closely follow Shelah [7, Section 1].

1. Introduction

Shelah [7] analyzed the question whether there are Borel sets in the plane which contain large squares but no perfect squares. A rank on models with...
a countable vocabulary was introduced and was used to define a cardinal \( \lambda_{\omega_1} \) (the first \( \lambda \) such that there is no model with universe \( \lambda \), countable vocabulary and rank \( < \omega_1 \)). It was shown in [7, Claim 1.12] that every Borel set \( B \subseteq \omega_2 \times \omega_2 \) which contains a \( \lambda_{\omega_1} \)-square must contain a perfect square. On the other hand, by [7, Theorem 1.13], if \( \mu = \mu^{\aleph_0} < \lambda_{\omega_1} \) then some ccc forcing notion forces that (the continuum is arbitrarily large and) some Borel set contains a \( \mu \)-square but no \( \mu^+ \)-square.

We would like to understand what the results mentioned above mean for general relations. Natural first step is to ask about Borel sets with \( \mu \geq \aleph_1 \) pairwise disjoint translations but without any perfect set of such translations, as motivated e.g. by Balcerzak, Roslanowski and Shelah [1] (we studied the \( \sigma \)-ideal of subsets of \( \omega_2 \) generated by Borel sets with a perfect set of pairwise disjoint translations) or Elekes and Keleti [3] (see Question 4.5 there). A generalization of this direction could follow Zakrzewski [8] who introduced perfectly \( k \)-small sets.

However, preliminary analysis of the problem revealed that another, somewhat orthogonal to the one described above, direction is more natural in the setting of [7]. Thus we investigate Borel sets with many, but not too many, pairwise overlapping intersections.

Easily, every uncountable Borel subset \( B \) of \( \omega_2 \) has a perfect set of pairwise non-disjoint translations (just consider a perfect set \( P \subseteq B \) and note that for \( x, y \in P \) we have \( 0, x+y \in (B+x) \cap (B+y) \)). The problem of many non-disjoint translations becomes more interesting if we demand that the intersections have more elements. Note that in \( \omega_2 \), if \( x+b_0 = y + b_1 \) then also \( x+b_1 = y + b_0 \), so \( x \neq y \) and \( |(B+x) \cap (B+y)| < \omega \) imply that \( |(B+x) \cap (B+y)| \) is even.

In the present paper we study the case when the intersections \( (B+x) \cap (B+y) \) have at least 6 elements. We show that for \( \lambda < \lambda_{\omega_1} \) there is a ccc forcing notion \( P \) adding a \( \Sigma^0_2 \) subset \( B \) of the Cantor space \( \omega_2 \) such that

- for some \( H \subseteq \omega_2 \) of size \( \lambda \), \( |(B+h) \cap (B+h')| \geq 6 \) for all \( h, h' \in H \), but

- for every perfect set \( P \subseteq \omega_2 \) there are \( x, x' \in P \) with \( |(B+x) \cap (B+x')| < 6 \).

We fully utilize the algebraic properties of \( (\omega_2, +) \), in particular the fact that all elements of \( \omega_2 \) are self-inverse.
In Section 2 of the paper we recall the rank from [7]. We give the relevant definitions, state and prove all the properties needed for our results later. In the third section we analyze when a $\Sigma^0_2$ subset of $\omega^2$ has a perfect set of pairwise overlapping translations. The main consistency result concerning adding a Borel set with no perfect set of overlapping translations is given in the fourth section.

**Notation.** Our notation is rather standard and compatible with that of classical textbooks (like Jech [4] or Bartoszyński and Judah [2]). However, in forcing we keep the older convention that a stronger condition is the larger one.

1. For a set $u$ we let
   \[ u^{(2)} = \{(x, y) \in u \times u : x \neq y\}. \]

2. The Cantor space $\omega^2$ of all infinite sequences with values 0 and 1 is equipped with the natural product topology and the group operation of coordinate-wise addition $+$ modulo 2.

3. Ordinal numbers will be denoted be the lower case initial letters of the Greek alphabet $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ as well as $\xi$. Finite ordinals (non-negative integers) will be denoted by letters $a, b, c, d, i, j, k, \ell, m, n, M$ and $\iota$.

4. The Greek letters $\kappa, \lambda$ will stand for uncountable cardinals.

5. For a forcing notion $\mathbb{P}$, all $\mathbb{P}$-names for objects in the extension via $\mathbb{P}$ will be denoted with a tilde below (e.g., $\tilde{\tau}, \tilde{X}$), and $\dot{G}_{\mathbb{P}}$ will stand for the canonical $\mathbb{P}$-name for the generic filter in $\mathbb{P}$.

### 2. The rank

We will remind some basic facts from [7, Section 1] concerning a rank (on models with countable vocabulary) which will be used in the construction of a forcing notion in the fourth section. For the convenience of the reader we provide proofs for most of the claims, even though they were given in [7]. Our rank $\text{rk}$ is the $\text{rk}^0$ of [7] and $\text{rk}^*$ is the $\text{rk}^2$ there.

In the setting of [7]. Thus we investigate Borel sets with many, but not too somewhat orthogonal to the one described above, direction is more natural
Let \( \lambda \) be a cardinal and \( \mathcal{M} \) be a model with the universe \( \lambda \) and a countable vocabulary \( \tau \).

**Definition 2.1.** 1. By induction on ordinals \( \delta \), for finite non-empty sets \( w \subseteq \lambda \) we define when \( \text{rk}(w, \mathcal{M}) \geq \delta \). Let \( w = \{\alpha_0, \ldots, \alpha_n\} \subseteq \lambda \), \( |w| = n + 1 \).

(a) \( \text{rk}(w) \geq 0 \) if and only if for every quantifier free formula \( \varphi \in \mathcal{L}(\tau) \) and each \( k \leq n \), if \( \mathcal{M} \models \varphi[\alpha_0, \ldots, \alpha_k, \ldots, \alpha_n] \) then the set

\[
\{ \alpha \in \lambda : \mathcal{M} \models \varphi[\alpha_0, \ldots, \alpha_{k-1}, \alpha, \alpha_{k+1}, \ldots, \alpha_n] \}
\]

is uncountable;

(b) if \( \delta \) is limit, then \( \text{rk}(w, \mathcal{M}) \geq \delta \) if and only if \( \text{rk}(w, \mathcal{M}) \geq \gamma \) for all \( \gamma < \delta \);

(c) \( \text{rk}(w, \mathcal{M}) \geq \delta + 1 \) if and only if for every quantifier free formula \( \varphi \in \mathcal{L}(\tau) \) and each \( k \leq n \), if \( \mathcal{M} \models \varphi[\alpha_0, \ldots, \alpha_k, \ldots, \alpha_n] \) then there is \( \alpha^* \in \lambda \setminus w \) such that

\[
\text{rk}(w \cup \{\alpha^*\}, \mathcal{M}) \geq \delta \text{ and } \mathcal{M} \models \varphi[\alpha_0, \ldots, \alpha_{k-1}, \alpha^*, \alpha_{k+1}, \ldots, \alpha_n].
\]

2. Similarly, for finite non-empty sets \( w \subseteq \lambda \) we define when \( \text{rk}^*(w, \mathcal{M}) \geq \delta \) (by induction on ordinals \( \delta \)). Let \( w = \{\alpha_0, \ldots, \alpha_n\} \subseteq \lambda \). We take clauses (a) and (b) above and

(c)* \( \text{rk}^*(w, \mathcal{M}) \geq \delta + 1 \) if and only if for every quantifier free formula \( \varphi \in \mathcal{L}(\tau) \) and each \( k \leq n \), if \( \mathcal{M} \models \varphi[\alpha_0, \ldots, \alpha_k, \ldots, \alpha_n] \) then there are pairwise distinct \( \langle \alpha^*_\zeta : \zeta < \omega_1 \rangle \subseteq \lambda \setminus \{w \setminus \{\alpha_k\}\} \) such that \( \alpha^*_0 = \alpha_k \) and for all \( \varepsilon < \zeta < \omega_1 \) we have

\[
\text{rk}^*(w \setminus \{\alpha_k\} \cup \{\alpha^*_\varepsilon, \alpha^*_\zeta\}, \mathcal{M}) \geq \delta
\]

and \( \mathcal{M} \models \varphi[\alpha_0, \ldots, \alpha_{k-1}, \alpha^*_\varepsilon, \alpha_{k+1}, \ldots, \alpha_n] \).

By a straightforward induction on \( \alpha \) one easily shows the following observation.

**Observation 2.2.** If \( \emptyset \neq v \subseteq w \) then

- \( \text{rk}(w, \mathcal{M}) \geq \delta \geq \gamma \) implies \( \text{rk}(v, \mathcal{M}) \geq \gamma \), and

- \( \text{rk}^*(w, \mathcal{M}) \geq \delta \geq \gamma \) implies \( \text{rk}^*(v, \mathcal{M}) \geq \gamma \).
Hence we may define the rank functions on finite non-empty subsets of $\lambda$.

**Definition 2.3.** The ranks $\text{rk}(w, M)$ and $\text{rk}^*(w, M)$ of a finite non-empty set $w \subseteq \lambda$ are defined as:

- $\text{rk}(w, M) = -1$ if $\neg(\text{rk}(w, M) \geq 0)$, and
  $\text{rk}^*(w, M) = -1$ if $\neg(\text{rk}^*(w, M) \geq 0)$,
- $\text{rk}(w, M) = \infty$ if $\text{rk}(w, M) \geq \delta$ for all ordinals $\delta$, and
  $\text{rk}^*(w, M) = \infty$ if $\text{rk}^*(w, M) \geq \delta$ for all ordinals $\delta$,
- for an ordinal $\delta$: $\text{rk}(w, M) = \delta$ if $\text{rk}(w, M) \geq \delta$ but $\neg(\text{rk}(w, M) \geq \delta + 1)$,
  and $\text{rk}^*(w, M) = \delta$ if $\text{rk}^*(w, M) \geq \delta$ but $\neg(\text{rk}^*(w, M) \geq \delta + 1)$.

**Definition 2.4.**

1. For an ordinal $\varepsilon$ and a cardinal $\lambda$ let $\text{NPr}_\varepsilon(\lambda)$ be the following statement: “there is a model $M^*$ with the universe $\lambda$ and a countable vocabulary $\tau^*$ such that $\sup\{\text{rk}(w, M^*) : \emptyset \neq w \in [\lambda]^{<\omega} \} < \varepsilon$.”

2. The statement $\text{NPr}_\varepsilon^*(\lambda)$ is defined similarly but using the rank $\text{rk}^*$.

3. $\text{Pr}_\varepsilon(\lambda)$ and $\text{Pr}_\varepsilon^*(\lambda)$ are the negations of $\text{NPr}_\varepsilon(\lambda)$ and $\text{NPr}_\varepsilon^*(\lambda)$, respectively.

**Observation 2.5.**

1. If a model $M^+$ (on $\lambda$) is an expansion\(^1\) of the model $M$, then $\text{rk}^*(w, M^+) \leq \text{rk}(w, M^+) \leq \text{rk}(w, M)$.

2. If $\lambda$ is uncountable and $\text{NPr}_\varepsilon(\lambda)$, then there is a model $M^*$ with the universe $\lambda$ and a countable vocabulary $\tau^*$ such that
   - $\text{rk}(\{\alpha\}, M^*) \geq 0$ for all $\alpha \in \lambda$ and
   - $\text{rk}(w, M^*) < \varepsilon$ for every finite non-empty set $w \subseteq \lambda$.

**Proposition 2.6** (See [7, Claim 1.7]).

1. $\text{NPr}_1(\omega_1)$.

2. If $\text{NPr}_\varepsilon(\lambda)$, then $\text{NPr}_{\varepsilon+1}(\lambda^+)$.

3. If $\text{NPr}_\varepsilon(\mu)$ for $\mu < \lambda$ and $\text{cf}(\lambda) = \omega$, then $\text{NPr}_{\varepsilon+1}(\lambda)$.

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\(^1\) So $M^+$ is a model with a countable vocabulary $\tau^* \supseteq \tau$, with the universe $\lambda$, and the interpretation of symbols from $\tau$ in $M^+$ is the same as in $M$. 
4. $\text{NPr}_\varepsilon(\lambda)$ implies $\text{NPr}_\varepsilon^*(\lambda)$.

**Proof.** (1) Let $Q$ be a binary relational symbol and let $M_1$ be a model with the universe $\omega_1$, the vocabulary $\tau(M_1) = \{Q\}$ and such that $Q^{M_1} = \{(\alpha, \beta) \in \omega_1 \times \omega_1 : \alpha < \beta\}$. Then for each $\alpha_0 < \alpha_1 < \omega_1$ we have $M_1 \models Q[\alpha_0, \alpha_1]$ but the set $\{\alpha < \omega_1 : M_1 \models Q[\alpha, \alpha_1]\}$ is countable. Hence $\text{rk}(w, M_1) = -1$ whenever $|w| \geq 2$ and $\text{rk}(\{\alpha\}, M_1) = 0$ for $\alpha \in \omega_1$. Consequently, $M_1$ witnesses $\text{NPr}_1(\omega_1)$.

(2) Assume $\text{NPr}_\varepsilon(\lambda)$ holds true as witnessed by a model $M$ with the universe $\lambda$ and a countable vocabulary $\tau$. We may assume that $\tau = \{R_i : i < \omega\}$, where each $R_i$ is a relational symbol of arity $n(i)$. Let $S$ be a new binary relational symbol, $T$ be a new unary relational symbol, and $Q_i$ be a new $(n(i) + 1)$-ary relational symbol (for $i < \omega$). Let $\tau^+ = \{R_i, Q_i : i < \omega\} \cup \{S, T\}$.

For each $\gamma \in [\lambda, \lambda^+]$ fix a bijection $f_\gamma : \gamma \rightarrow \lambda$. We define a model $M^+$:

- the vocabulary of $M^+$ is $\tau^+$ and the universe of $M^+$ is $\lambda^+$,
- $R_i^{M^+} = R_i^M \subseteq \lambda^{n(i)}$,
- $Q^i_{M^+} = \{(\alpha_0, \ldots, \alpha_{n(i)-1}, \alpha_{n(i)}): \lambda \leq \alpha_{n(i)} < \lambda^+ & (\forall \ell < n(i))(\alpha_{\ell} < \alpha_{n(i)}) \land (f_{\alpha_{n(i)}}(\alpha_0), \ldots, f_{\alpha_{n(i)}}(\alpha_{n(i)-1})) \in R_i^M\}$,
- $S^{M^+} = \{((\alpha_0, \alpha_1) \in \lambda^+ \times \lambda^+: \alpha_0 < \alpha_1\}$ and $T^{M^+} = [\lambda, \lambda^+]$.

**Claim 2.6.1.** (i) If $\lambda \leq \gamma < \lambda^+$, $\emptyset \neq w \subseteq \gamma$, then $\text{rk}(w \cup \{\gamma\}, M^+) \leq \text{rk}(f_\gamma[w], M)$ and thus $\text{rk}(w \cup \{\gamma\}, M^+) < \varepsilon$.

(ii) If $\emptyset \neq w \subseteq \lambda$, then $\text{rk}(w, M^+) \leq \text{rk}(w, M)$ and thus $\text{rk}(w, M^+) < \varepsilon$.

(iii) If $\lambda \leq \gamma < \lambda^+$, then $\text{rk}(\{\gamma\}, M^+) \leq \varepsilon$.

**Proof of the Claim.** (i) By induction on $\alpha$ we show that $\alpha \leq \text{rk}(w \cup \{\gamma\}, M^+)$ implies $\alpha \leq \text{rk}(f_\gamma[w], M)$ (for all sets $w \subseteq \gamma$ with fixed $\gamma \in [\lambda, \lambda^+]$).

(*) Assume $\text{rk}(w \cup \{\gamma\}, M^+) \geq 0$, $w = \{\alpha_0, \ldots, \alpha_n\}$ and $k \leq n$. Let $\phi(x_0, \ldots, x_n)$ be a quantifier free formula in the vocabulary $\tau$ such that $M \models \phi[f_\gamma(\alpha_0), \ldots, f_\gamma(\alpha_k), \ldots, f_\gamma(\alpha_n)]$. 
Let \( \varphi^*(x_0, \ldots, x_n, x_{n+1}) \) be a quantifier free formula in the vocabulary \( \tau^+ \) obtained from \( \varphi \) by replacing each \( R_i(y_0, \ldots, y_{n(i)}-1) \) (where \( \{y_0, \ldots, y_{n(i)}-1\} \subseteq \{x_0, \ldots, x_n\} \)) with \( Q_i(y_0, \ldots, y_{n(i)}-1, x_{n+1}) \) and let \( \varphi^+ \) be

\[
\varphi^*(x_0, \ldots, x_n, x_{n+1}) \land S(x_0, x_{n+1}) \land \ldots \land S(x_n, x_{n+1}).
\]

Then \( M^+ \models \varphi^+[\alpha_0, \ldots, \alpha_k, \ldots, \alpha_n, \gamma] \). By our assumption on \( w \cup \{\gamma\} \) we know that the set

\[
A = \{ \beta < \lambda^+ : M^+ \models \varphi^+[\alpha_0, \ldots, \alpha_{k-1}, \beta, \alpha_{k+1}, \ldots, \alpha_n, \gamma] \}
\]

is uncountable. Clearly \( A \subseteq \gamma \) (note \( S(x_k, x_{n+1}) \) in \( \varphi^+ \)) and thus the set \( f_\gamma[A] \) is an uncountable subset of \( \lambda \). For each \( \beta \in A \) we have

\[
M \models \varphi[f_\gamma(\alpha_0), \ldots, f_\gamma(\beta), \ldots, f_\gamma(\alpha_n)],
\]

so now we may conclude that \( \operatorname{rk}(f_\gamma[w], M) \geq 0 \).

(\ast)_1 Assume \( \operatorname{rk}(w \cup \{\gamma\}, M^+) \geq \alpha + 1 \). Let \( \varphi(x_0, \ldots, x_n) \) be a quantifier free formula in the vocabulary \( \tau, k \leq n \) and \( w = \{\alpha_0, \ldots, \alpha_n\} \), and suppose that \( M \models \varphi[f_\gamma(\alpha_0), \ldots, f_\gamma(\alpha_k), \ldots, f_\gamma(\alpha_n)] \). Let \( \varphi^* \) and \( \varphi^+ \) be defined exactly as in (\ast)_0. Then \( M^+ \models \varphi^+[\alpha_0, \ldots, \alpha_k, \ldots, \alpha_n, \gamma] \). By our assumption there is \( \beta^* \in \lambda^+ \setminus (w \cup \{\gamma\}) \) such that \( M^+ \models \varphi^+[\alpha_0, \ldots, \beta^*, \ldots, \alpha_n, \gamma] \) and \( \operatorname{rk}(w \cup \{\gamma, \beta^*\}, M^+) \geq \alpha \). Necessarily \( \beta^* < \gamma \), and by the inductive hypothesis \( \operatorname{rk}(f_\gamma[w \cup \{\beta^*\}], M) \geq \alpha \). Clearly \( M \models \varphi[f_\gamma(\alpha_0), \ldots, f_\gamma(\beta^*), \ldots, f_\gamma(\alpha_n)] \)

and we may conclude \( \operatorname{rk}(f_\gamma[w], M) \geq \alpha + 1 \).

(\ast)_2 If \( \alpha \) is limit and \( \operatorname{rk}(w \cup \{\gamma\}, M^+) \geq \alpha \) then, by the inductive hypothesis, for each \( \beta < \alpha \) we have \( \beta \leq \operatorname{rk}(w \cup \{\gamma\}, M^+) \leq \operatorname{rk}(f_\gamma[w], M) \). Hence \( \alpha \leq \operatorname{rk}(f_\gamma[w], M) \).

(ii) Induction similar to part (i). For a quantifier free formula \( \varphi(x_0, \ldots, x_n) \) in the vocabulary \( \tau \), let \( \varphi^* \) be the formula \( \varphi(x_0, \ldots, x_n) \land \neg T(x_0) \land \ldots \land \neg T(x_n) \) (so \( \varphi^* \) is a quantifier free formula in the vocabulary \( \tau^+ \)). If \( \varphi \) witnesses that \( \neg(\operatorname{rk}(w, M) \geq 0) \), then \( \varphi^* \) witnesses \( \neg(\operatorname{rk}(w, M^+) \geq 0) \), and similarly with \( \alpha + 1 \) in place of 0.

(iii) Suppose towards contradiction that \( \varepsilon + 1 \leq \operatorname{rk}(\{\gamma\}, M^+) \). Since \( M^+ \models T[\gamma] \), we may find \( \gamma' \neq \gamma \) such that \( \operatorname{rk}(\{\gamma, \gamma'\}, M^+) \geq \varepsilon \) and \( M^+ \models T[\gamma'] \). Let \( \{\gamma, \gamma'\} = \{\gamma_0, \gamma_1\} \) where \( \gamma_0 < \gamma_1 \). It follows from part (i) that \( \operatorname{rk}(\{\gamma_0, \gamma_1\}, M^+) < \varepsilon \), a contradiction. \( \square \)
It follows from Claim 2.6.1 (and Observation 2.2) that \( \text{rk}(w, M^+) \leq \varepsilon \) for every non-empty set \( w \subseteq \lambda^+ \). Consequently, the model \( M^+ \) witnesses \( \text{NP}_{\varepsilon+1}(\lambda^+) \).

(3) Let \( \langle \mu_n : n < \omega \rangle \) be an increasing sequence cofinal in \( \lambda \). For each \( n \) fix a model \( M_n \) with a countable vocabulary \( \tau(M_n) \) consisting of relational symbols only and with the universe \( \mu_n \) and such that \( \text{rk}(w, M_n) < \varepsilon \) for nonempty finite \( w \subseteq \mu_n \). We also assume that \( \tau(M_n) \cap \tau(M_{m}) = \emptyset \) for \( n < m < \omega \). Let \( P_n \) (for \( n < \omega \)) be new unary relational symbols and let \( \tau = \bigcup \{ \tau(M_n) : n < \omega \} \cup \{ P_n : n < \omega \} \). Consider a model \( M \) in vocabulary \( \tau \) with the universe \( \lambda \) and such that

- \( P^M_n = \mu_n \) for \( n < \omega \), and
- for each \( n < \omega \) and \( S \in \tau(M_n) \) we have \( S^M = S^{M_n} \).

**Claim 2.6.2.** If \( w \) is a finite non-empty subset of \( \mu_n \), \( n < \omega \), then \( \text{rk}(w, M) \leq \text{rk}(w, M_n) < \varepsilon \).

**Proof of the Claim.** Similar to the proofs in Claim 2.6.1. \( \square \)

(4) Follows from Observation 2.5(1). \( \square \)

**Proposition 2.7.** (See [7, Conclusion 1.8]). Assume \( \beta < \alpha < \omega_1 \), \( M \) is a model with a countable vocabulary \( \tau \) and the universe \( \mu \), \( m, n < \omega \), \( n > 0 \), \( A \subseteq \mu \) and \( |A| \geq \beth_\omega \). Then there is \( w \subseteq A \) with \( |w| = n \) and \( \text{rk}^*(w, M) \geq \omega \cdot \beta + m \). \(^2\)

**Proof.** Induction on \( \alpha < \omega_1 \).

**Step \( \alpha = 1 \) (and \( \beta = 0 \)):** Let \( M, \mu, n, m \) be as in the assumptions, \( A \subseteq \mu \) and \( |A| \geq \beth_\omega \). Using the Erdős–Rado theorem we may choose a sequence \( \langle \alpha_\varepsilon : \varepsilon < \omega_2 \rangle \) of distinct elements of \( A \) such that:

(a) the quantifier free type of \( \langle \alpha_{\varepsilon_0}, \ldots, \alpha_{\varepsilon_{m+n}} \rangle \) in \( M \) is constant for \( \varepsilon_0 < \ldots < \varepsilon_{m+n} < \omega_2 \), and

(b) for each \( k \leq m + n \) the value of \( \min\{ \omega, \text{rk}^*(\{ \alpha_{\varepsilon_0}, \ldots, \alpha_{\varepsilon_{m+n-k}} \}, M) \} \) is constant for \( \varepsilon_0 < \ldots < \varepsilon_{m+n-k} < \omega_2 \).

\(^2\) "\( \cdot \)" stands for the ordinal multiplication.
Let $\zeta_\ell = \omega_1 \cdot (\ell + 1)$ (for $\ell = -1, 0, \ldots, m+n$). Suppose $\phi(x_0, \ldots, x_{m+n}) \in \mathcal{L}(\tau)$ is a quantifier free formula, $k \leq m+n$ and
\[ M \models \phi[\alpha_{\zeta_0}, \ldots, \alpha_{\zeta_k}, \ldots, \alpha_{\zeta_{m+n}}]. \]

It follows from the property stated in (a) above that for every $\varepsilon$ in the (uncountable) interval $(\zeta_{k-1}, \zeta_k)$ we have
\[ M \models \varphi[\alpha_{\zeta_0}, \ldots, \alpha_{\zeta_{k-1}}, \alpha_{\varepsilon}, \alpha_{\zeta_{k+1}}, \ldots, \alpha_{\zeta_{m+n}}]. \]

Consequently, $\text{rk}^*(\{\alpha_{\zeta_0}, \ldots, \alpha_{\zeta_{m+n}}\}, M) \geq 0$, and the homogeneity stated in (b) implies that for every nonempty set $w \subseteq \omega_2$ with at most $m+n+1$ elements we have $\text{rk}^*(\{\alpha_{\varepsilon} : \varepsilon \in w\}, M) \geq 0$. Now, by induction on $k \leq m+n$ we will argue that

\[(*)_k \text{ for every nonempty set } w \subseteq \omega_2 \text{ with at most } m+n+1-k \text{ elements } \quad \text{we have } \text{rk}^*(\{\alpha_{\varepsilon} : \varepsilon \in w\}, M) \geq k. \]

We have already justified $(*)_0$. For the inductive step assume $(*)_k$ and $k < m+n$. Let $\zeta_\ell = \omega_1 \cdot (\ell + 1)$ and suppose that $\varphi(x_0, \ldots, x_{m+n-k-1})$ is a quantifier free formula, $M \models \varphi[\alpha_{\zeta_0}, \ldots, \alpha_{\zeta_z}, \ldots, \alpha_{\zeta_{m+n-k-1}}]$ and $0 \leq z \leq m+n-k-1$. By the homogeneity stated in (a), for every $\varepsilon$ in the uncountable interval $(\zeta_{z-1}, \zeta_z)$ we have
\[ M \models \varphi[\alpha_{\zeta_0}, \ldots, \alpha_{\zeta_{z-1}}, \alpha_{\varepsilon}, \alpha_{\zeta_{z+1}}, \ldots, \alpha_{\zeta_{m+n-k-1}}]. \]

The inductive hypothesis $(*)_k$ implies that
\[ \text{rk}^*(\{\alpha_{\zeta_0}, \ldots, \alpha_{\zeta_{z-1}}, \alpha_{\varepsilon}, \alpha_{\zeta_{z+1}}, \ldots, \alpha_{\zeta_{m+n-k-1}}\}, M) \geq k \]
(for any $\zeta_{z-1} < \varepsilon < \zeta_z \leq \zeta_z$). Now we easily conclude that $k+1 \leq \text{rk}^*(\{\alpha_{\zeta_0}, \ldots, \alpha_{\zeta_{m+n-k-1}}\}, M)$ and $(*)_k+1$ follows by the homogeneity given by (b).

Finally note that $(*)_m+1$ gives the desired conclusion: taking any $\varepsilon_0 < \ldots < \varepsilon_{n-1} < \omega_2$ we will have $m+1 \leq \text{rk}^*(\{\alpha_{\varepsilon_0}, \ldots, \alpha_{\varepsilon_{n-1}}\}, M)$.

**Step $\alpha = \gamma + 1$:** Let $M, \mu, n, m$ be as in the assumptions, $A \subseteq \mu$ and $|A| \geq \beth_{\omega \cdot \gamma+\omega}$. By the Erdős–Rado theorem we may choose a sequence $\langle \alpha_{\varepsilon} : \varepsilon < \beth_{\omega \cdot \gamma} \rangle$ of distinct elements of $A$ such that the following two demands are satisfied.
(c) The quantifier free type of $\langle \alpha_{\varepsilon_0}, \ldots, \alpha_{\varepsilon_{m+n}} \rangle$ in $M$ is constant for $\varepsilon_0 < \ldots < \varepsilon_{m+n} < \omega$. 

(d) For each $k \leq m+n$ the value of $\min\{\omega \cdot (\gamma+1), \text{rk}^*\{\alpha_{\varepsilon_0}, \ldots, \alpha_{\varepsilon_{m+n-k}}\}, M\}$ is constant for $\varepsilon_0 < \ldots < \varepsilon_{m+n-k} < \omega$. 

For any $\ell < \omega$ and $\gamma' < \gamma$, we may apply the inductive hypothesis to $\{\alpha_{\varepsilon} : \varepsilon < \omega\}, \ell, m+n+1$ and $\gamma'$ to find $\varepsilon_0 < \ldots < \varepsilon_{m+n} < \omega$ such that $\text{rk}^*\{\alpha_{\varepsilon_0}, \ldots, \alpha_{\varepsilon_{m+n}}\}, M) \geq \omega \cdot \gamma' + \ell$. By the homogeneity in (d) this implies that

$$(**)_0 \text{ for all } \varepsilon_0 < \ldots < \varepsilon_{m+n} < \omega \text{ we have } \text{rk}^*\{\alpha_{\varepsilon_0}, \ldots, \alpha_{\varepsilon_{m+n}}\}, M) \geq \omega \cdot \gamma.$$ 

Now, by induction on $k \leq m+n$ we argue that

$$(**)_k \text{ for each } \varepsilon_0 < \ldots < \varepsilon_{m+n-k} < (\omega) \text{ we have } \omega \cdot \gamma + k \leq \text{rk}^*\{\alpha_{\varepsilon_0}, \ldots, \alpha_{\varepsilon_{m+n-k}}\}, M).$$ 

So assume $(**)_k$, $k < m+n$ and let $\zeta_\ell = \omega_1 \cdot (\ell+1)$ (for $\ell = -1, 0, \ldots, m+n$) and $0 \leq z \leq m+n-k-1$. Suppose that $M \models \varphi[\alpha_{\zeta_0}, \ldots, \alpha_{\zeta_z}, \ldots, \alpha_{\zeta_{m+n-k-1}}]$. Then by the homogeneity in (c), for every $\varepsilon$ in the uncountable interval $(\zeta_{z-1}, \zeta_z)$ we have $M \models \varphi[\alpha_{\zeta_0}, \ldots, \alpha_{\zeta_{z-1}}, \alpha_\varepsilon, \alpha_{\zeta_z+1}, \ldots, \alpha_{\zeta_{m+n-k-1}}]$. By the inductive hypothesis $(**)_k$ we know

$$\omega \cdot \gamma + k \leq \text{rk}^*\{\alpha_{\zeta_0}, \ldots, \alpha_{\zeta_{z-1}}, \alpha_\varepsilon, \alpha_{\zeta_z+1}, \ldots, \alpha_{\zeta_{m+n-k-1}}\}, M)$$ 

(for $\zeta_{z-1} < \varepsilon < \zeta_z$). Now we easily conclude that $\omega \cdot \gamma + k + 1 \leq \text{rk}^*\{\alpha_{\zeta_0}, \ldots, \alpha_{\zeta_{m+n-k-1}}\}, M)$, and $(**)_{k+1}$ follows by the homogeneity in (d).

Finally note that $(**)_m+1$ gives the desired conclusion: taking any $\zeta_0 < \ldots < \zeta_{n-1} < \omega$ we will have $\text{rk}^*\{\alpha_{\zeta_0}, \ldots, \alpha_{\zeta_{n-1}}\}, M) \geq \omega \cdot \gamma + m + 1$.

**Step $\alpha$ is limit:** Straightforward. $\square$

**Definition 2.8.** Let $\lambda_{\omega_1}$ be the smallest cardinal $\lambda$ such that $\Pr_{\omega_1}(\lambda)$ and $\lambda^*_{\omega_1}$ be the smallest cardinal $\lambda$ such that $\Pr^*_{\omega_1}(\lambda)$.

**Corollary 2.9.** 1. If $\alpha < \omega_1$, then $\text{NPr}_{\omega_1}(N_\alpha)$. 
2. \( \Pr_{\omega_1}^*(\mathfrak{A}_{\omega_1}) \) holds and hence also \( \Pr_{\omega_1}(\mathfrak{A}_{\omega_1}) \).

3. \( n_{\omega_1} \leq \lambda_{\omega_1} \leq \lambda_{\omega_1}^* \leq \mathfrak{A}_{\omega_1}. \)

**Proof.** (1) Immediately from Proposition 2.6, by induction on \( \alpha < \omega_1. \)
(2) Follows from Proposition 2.7 (and 2.6(4)).
(3) By clauses (1), (2) above. \( \square \)

**Proposition 2.10.** (See [7, Claim 1.10(1)].) If \( \mathbb{P} \) is a ccc forcing notion and \( \lambda \) is a cardinal such that \( \Pr_{\omega_1}^*(\lambda) \) holds, then \( \mathbb{P} \Vdash " \Pr_{\omega_1}^*(\lambda) \) and hence also \( \Pr_{\omega_1}(\lambda) " \).

**Proof.** Suppose towards contradiction that for some \( p \in \mathbb{P} \) we have \( p \Vdash \text{NPr}_{\omega_1}^*(\lambda) \). Let \( \tau = \{R_{n,\zeta} : n, \zeta < \omega \} \) where \( R_{n,\zeta} \) is an \( n \)-ary relation symbol (for \( n, \zeta < \omega \)). Then we may pick a name \( \mathbb{M} \) for a model on \( \lambda \) in vocabulary \( \tau \) and an ordinal \( \alpha_0 < \omega_1 \) such that

\[
p \Vdash "\mathbb{M} = (\lambda, \{R_{n,\zeta}^{\mathbb{M}}\}_{n,\zeta<\omega}) \text{ is a model such that}
\]
(a) for every \( n \) and a quantifier free formula \( \varphi(x_0, \ldots, x_{n-1}) \in \mathcal{L}(\tau) \)
there is \( \zeta < \omega \) such that for all \( \gamma_0, \ldots, \gamma_{n-1} \)
\( \mathbb{M} \models \varphi[\gamma_0, \ldots, \gamma_{n-1}] \iff R_{n,\zeta}[\gamma_0, \ldots, \gamma_{n-1}] \)
(b) \( \sup\{\text{rk}(w, \mathbb{M}) : \emptyset \neq w \in [\lambda]^{<\omega}) < \alpha_0 \". \)

Now, let \( S_{n,\zeta,\beta,k} \) be an \( n \)-ary predicate (for \( k < n, \zeta < \omega \) and \( -1 \leq \beta < \alpha_0 \)) and let \( \tau^* = \{S_{n,\zeta,\beta,k} : k < n < \omega, \zeta < \omega \) and \( -1 \leq \beta < \alpha_0 \}. \) (So \( \tau^* \) is a countable vocabulary.) We define a model \( \mathbb{M}^* \) in the vocabulary \( \tau^* \). The universe of \( \mathbb{M}^* \) is \( \lambda \) and for \( k < n, \zeta < \omega \) and \( -1 \leq \beta < \alpha_0 \):

\[
S_{n,\zeta,\beta,k}^{\mathbb{M}^*} = \{(\gamma_0, \ldots, \gamma_{n-1}) \in n\lambda : \gamma_0 < \ldots < \gamma_{n-1} \text{ and}
\text{some condition } q \geq p \text{ forces that}
\text{"} \mathbb{M} \models R_{n,\zeta}[\gamma_0, \ldots, \gamma_{n-1}] \text{ and } \text{rk}^*(\{\gamma_0, \ldots, \gamma_{n-1}\}, \mathbb{M}) = \beta \text{ and}
R_{n,\zeta,\beta,k} \text{ witness that } \neg(\text{rk}^*(\{\gamma_0, \ldots, \gamma_{n-1}\}, \mathbb{M}) \geq \beta + 1) " \}.
\]

**Claim 2.10.1.** For every \( n \) and every increasing tuple \( (\gamma_0, \ldots, \gamma_{n-1}) \in n\lambda \) there are \( \zeta < \omega \) and \( -1 \leq \beta < \alpha_0 \) and \( k < n \) such that \( \mathbb{M}^* \models S_{n,\zeta,\beta,k}[\gamma_0, \ldots, \gamma_{n-1}] \).

**Proof of the Claim.** Clear. \( \square \)
Claim 2.10.2. If \((\gamma_0, \ldots, \gamma_{n-1}) \in \lambdaint{\lambda} \text{ and } M^* \models S_{n, \zeta, \beta, k}[\gamma_0, \ldots, \gamma_{n-1}]\),
then
\[ \text{rk}^*(\{\gamma_0, \ldots, \gamma_{n-1}\}, M^*) \leq \beta. \]

**Proof of the Claim.** First let us deal with the case of \(\beta = -1\). Assume towards contradiction that \(M^* \models S_{n, \zeta, -1, k}[\gamma_0, \ldots, \gamma_{n-1}]\), but \(\text{rk}^*(\{\gamma_0, \ldots, \gamma_{n-1}\}, M^*) \geq 0\). Then we may find distinct \(\langle \delta_\varepsilon : \varepsilon < \omega_1 \rangle \subseteq \lambda \setminus \{\gamma_0, \ldots, \gamma_{n-1}\}\) such that

\[(\otimes)_1 \ M^* \models S_{n, \zeta, -1, k}[\gamma_0, \ldots, \gamma_{k-1}, \delta_\varepsilon, \gamma_{k+1}, \ldots, \gamma_{n-1}] \text{ for all } \varepsilon < \omega_1.\]

For \(\varepsilon < \omega_1\) let \(p_\varepsilon \in \mathbb{P}\) be such that \(p_\varepsilon \geq p\) and

\[ p_\varepsilon \models " M \models R_{n, \zeta}[\gamma_0, \ldots, \delta_\varepsilon, \ldots, \gamma_{n-1}] \text{ and } \text{rk}^*(\{\gamma_0, \ldots, \delta_\varepsilon, \ldots, \gamma_{n-1}\}, M) = -1 \text{ and } R_{n, \zeta, k} \text{ witness that } \neg (\text{rk}^*(\{\gamma_0, \ldots, \gamma_{k-1}, \delta_\varepsilon, \gamma_{k+1}, \ldots, \gamma_{n-1}\}, M) \geq 0) " \]

Let \(Y\) be a name \(\mathbb{P}\)-name such that \(p \models Y = \{\varepsilon < \omega_1 : p_\varepsilon \in \mathcal{G}_\mathbb{P}\}\). Since \(\mathbb{P}\) satisfies ccc, we may pick \(p^* \geq p\) such that \(p^* \models "Y \text{ is uncountable}"\). Since

\[ p^* \models (\forall \varepsilon \in Y)(M \models R_{n, \zeta}[\gamma_0, \ldots, \gamma_{k-1}, \delta_\varepsilon, \gamma_{k+1}, \ldots, \gamma_{n-1}]), \]

then also

\[ p^* \models \{\delta < \lambda : M \models R_{n, \zeta}[\gamma_0, \ldots, \gamma_{k-1}, \delta, \gamma_{k+1}, \ldots, \gamma_{n-1}]\} \text{ is uncountable.} \]

But

\[ p^* \models (\forall \varepsilon \in Y) \]

\[ (R_{n, \zeta, k} \text{ witness } \neg (\text{rk}^*(\{\gamma_0, \ldots, \gamma_{k-1}, \delta_\varepsilon, \gamma_{k+1}, \ldots, \gamma_{n-1}\}, M) \geq 0)), \]

and hence

\[ p^* \models \{\delta < \lambda : M \models R_{n, \zeta}[\gamma_0, \ldots, \gamma_{k-1}, \delta, \gamma_{k+1}, \ldots, \gamma_{n-1}]\} \text{ is countable,} \]
a contradiction.

Next we continue the proof of the Claim by induction on \(\beta < \alpha_0\), so we assume that \(0 \leq \beta\) and for \(\beta' < \beta\) our claim holds true (for any \(n, \zeta, k\)). Assume towards contradiction that \(M^* \models S_{n, \zeta, \beta, k}[\gamma_0, \ldots, \gamma_{n-1}]\), but \(\text{rk}^*(\{\gamma_0, \ldots, \gamma_{n-1}\}, M^*) \geq \beta + 1\). Then we may find distinct \(\langle \delta_\varepsilon : \varepsilon < \omega_1 \rangle \subseteq \lambda \setminus (w \setminus \{\gamma_k\})\) such that
so we assume that $0 \leq \lambda$ but $\text{rk}^{\ast} (\{ \gamma_0, \ldots, \gamma_{k-1}, \delta_{\varepsilon}, \gamma_k+1, \ldots, \gamma_{n-1} \}, M^\ast) \geq \beta$ for all $\varepsilon < \zeta < \omega_1$. For $\varepsilon < \omega_1$ let $p_\varepsilon \in \mathbb{P}$ be such that $p_\varepsilon \geq p$ and

$\begin{align*}
p_\varepsilon &\models " M \models R_{n, \zeta}[\gamma_0, \ldots, \gamma_{k-1}, \delta_{\varepsilon}, \gamma_k+1, \ldots, \gamma_{n-1}] \\
&\text{and } \text{rk}^{\ast}(\{ \gamma_0, \ldots, \gamma_{k-1}, \delta_{\varepsilon}, \gamma_k+1, \ldots, \gamma_{n-1} \}, M) = \beta \\
&\text{and } R_{n, \zeta}, k \text{ witness that } \\
&\neg (\text{rk}^{\ast}(\{ \gamma_0, \ldots, \gamma_{k-1}, \delta_{\varepsilon}, \gamma_k+1, \ldots, \gamma_{n-1} \}, M) \geq \beta + 1)"
\end{align*}$

Take $p^\ast \geq p$ such that

$p^\ast \models " Y \overset{\text{def}}{=} \{ \varepsilon < \omega_1 : p_\varepsilon \in G_\mathbb{P} \text{ is uncountable} \}.$

Since

$p^\ast \models (\forall \varepsilon \in Y) \left( M \models R_{n, \zeta}[\gamma_0, \ldots, \gamma_{k-1}, \delta_{\varepsilon}, \gamma_k+1, \ldots, \gamma_{n-1}] \wedge \\
R_{n, \zeta}, k \text{ witness that } \neg (\text{rk}^{\ast}(\{ \gamma_0, \ldots, \gamma_{k-1}, \delta_{\varepsilon}, \gamma_k+1, \ldots, \gamma_{n-1} \}, M) \geq \beta + 1) \right),$ 

we see that

$p^\ast \not\models (\forall \varepsilon, \zeta \in Y)(\varepsilon \neq \zeta \Rightarrow \text{rk}^{\ast}(\{ \gamma_0, \ldots, \gamma_{k-1}, \delta_{\varepsilon}, \delta_{\zeta}, \gamma_k+1, \ldots, \gamma_{n-1} \}, M) \geq \beta).$

Consequently we may pick $q \geq p^\ast, \varepsilon_0, \zeta_0 < \omega_1$ and $\gamma < \beta$ and $\xi < \omega$ and $\ell \leq n$ such that $\delta_{\varepsilon_0} < \delta_{\zeta_0}$ and

$q \models " p_{\varepsilon_0}, p_{\zeta_0} \in G_\mathbb{P} \text{ and } \text{rk}^{\ast}(\{ \gamma_0, \ldots, \gamma_{k-1}, \delta_{\varepsilon_0}, \delta_{\zeta_0}, \gamma_k+1, \ldots, \gamma_{n-1} \}, M) = \gamma \\
\text{and } R_{n+1, \xi} \text{ and } \ell \text{ witness that } \\
\neg (\text{rk}^{\ast}(\{ \gamma_0, \ldots, \gamma_{k-1}, \delta_{\varepsilon_0}, \delta_{\zeta_0}, \gamma_k+1, \ldots, \gamma_{n-1} \}, M) \geq \gamma + 1)\".$

Then $M^\ast \models S_{n+1, \xi, \gamma, \ell}[\gamma_0, \ldots, \gamma_{k-1}, \delta_{\varepsilon_0}, \delta_{\zeta_0}, \gamma_k+1, \ldots, \gamma_{n-1}]$ and by the inductive hypothesis $\text{rk}^{\ast}(\{ \gamma_0, \ldots, \gamma_{k-1}, \delta_{\varepsilon_0}, \delta_{\zeta_0}, \gamma_k+1, \ldots, \gamma_{n-1} \}, M) \leq \gamma$, contradicting clause $(\oplus)_2$ above.

\begin{flushright} $\square$ \end{flushright}

**Corollary 2.11.** Let $\mu = \beth_{\omega_1} \leq \kappa$ and $\mathbb{C}_\kappa$ be the forcing notion adding $\kappa$ Cohen reals. Then $\models_{\mathbb{C}_\kappa} \lambda_{\omega_1} \leq \mu \leq \mathfrak{c}$. 
3. Spectrum of translation non-disjointness

Definition 3.1. Let $B \subseteq \omega^2$ and $1 \leq \kappa \leq \mathfrak{c}$.

1. We say that $B$ is perfectly orthogonal to $\kappa$-small (or a $\kappa$-pots-set) if there is a perfect set $P \subseteq \omega^2$ such that $|(B + x) \cap (B + y)| \geq \kappa$ for all $x, y \in P$. The set $B$ is a $\kappa$-npots-set if it is not $\kappa$-pots.

2. We say that $B$ has $\lambda$ many pairwise $\kappa$-nondisjoint translations if for some set $X \subseteq \omega^2$ of cardinality $\lambda$, for all $x, y \in X$ we have $|(B + x) \cap (B + y)| \geq \kappa$.

3. We define the spectrum of translation $\kappa$-non-disjointness of $B$ as

$$\text{stnd}_\kappa(B) = \{(x, y) \in \omega^2 \times \omega^2 : |(B + x) \cap (B + y)| \geq \kappa\}.$$ 

Remark 3.2. 1. Note that if $B \subseteq \omega^2$ is an uncountable Borel set, then there is a perfect set $P \subseteq B$. For $B, P$ as above for every $x, y \in P$ we have $0 = x + x = y + y \in (B + x) \cap (B + y)$ and $x + y \in (B + x) \cap (B + y)$. Consequently every uncountable Borel subset of $\omega^2$ is a 2-pots-set.

2. Assume $B \subseteq \omega^2$ and $x, y \in \omega^2$. If $b_x, b_y \in B$ and $b_x + x = b_y + y \in (B + x) \cap (B + y)$, then also $b_x + y = b_y + x \in (B + x) \cap (B + y)$. Consequently, if $(B + x) \cap (B + y) \neq \emptyset$ is finite, then it has an even number of elements.

Proposition 3.3. 1. Let $1 \leq \kappa \leq \mathfrak{c}$. A set $B \subseteq \omega^2$ is a $\kappa$-pots-set if and only if there is a perfect set $P \subseteq \omega^2$ such that $P \times P \subseteq \text{stnd}_\kappa(B)$.

2. Assume $k < \omega$. If $B$ is $\Sigma^0_2$, then $\text{stnd}_k(B)$ is $\Sigma^0_2$ as well. If $B$ is Borel, then $\text{stnd}_k(B)$ and $\text{stnd}_\omega(B)$ are $\Sigma^1_1$ and $\text{stnd}_\omega(B)$ is $\Delta^1_2$.

3. Let $\mathfrak{c} < \lambda \leq \mu$ and let $\mathbb{C}_\mu$ be the forcing notion adding $\mu$ Cohen reals. Then, remembering Definition 3.1(2),

$$\Vdash_{\mathbb{C}_\mu} \text{"if a Borel set } B \subseteq \omega^2 \text{ has } \lambda \text{ many pairwise } \kappa\text{-non-disjoint translates, then } B \text{ is a } \kappa\text{-pots-set".}$$
4. If $k < \omega$, $B$ is a (code for) $\Sigma^0_2$ \textit{k−pots−set} and $\mathbb{P}$ is a forcing notion, then $\Vdash_{\mathbb{P}} \text{"B is a (code for) k−pots−set"}$. 

5. Assume $\text{Pr}_{\omega_1}(\lambda)$. If $\kappa \leq \omega$ and a Borel set $B \subseteq \omega^2$ has $\lambda$ many pairwise $\kappa$–nondisjoint translates, then it is a $\kappa$–\textit{pots−set}.

\textbf{Proof.} (2) Let $B = \bigcup_{n<\omega} F_n$, where each $F_n$ is a closed subset of $\omega^2$. Then 

\[(x,y) \in \text{std}_k(B) \iff \left( \exists n_0, \ldots, n_k-1, m_0, \ldots, m_k-1, N < \omega \right) \left( \exists z_0, \ldots, z_k \in \omega^2 \right) \left( \forall i, j < k \right) \left( i \neq j \implies z_i | N \neq z_j | N \right) \land z_i + x \in F_{n_i} \land z_i + y \in F_{m_i} \right) \]

The formula

\[\left( \forall i, j < k \right) \left( (i \neq j \implies z_i | N \neq z_j | N) \land z_i + x \in F_{n_i} \land z_i + y \in F_{m_i} \right) \]

represents a compact subset of $(\omega^2)^{k+2}$ and hence easily the assertion follows.

(3) This is a consequence of (1,2) above and Shelah [7, Fact 1.16].

(4) If $B$ is a $\Sigma^0_2$ set then the formula “there is a perfect set $P \subseteq \omega^2$ such that for all $x, y \in P$ we have $(x,y) \in \text{std}_k(B)$ ” is $\Sigma^1_2$ (remember (2) above).

(5) By [7, Claim 1.12(1)]. \hfill \Box

We want to analyze $k$–\textit{pots−sets} in more detail, restricting ourselves to $\Sigma^0_2$ subsets of $\omega^2$ and even $k < \omega$. For the rest of this section we assume the following Hypothesis.

\textbf{Hypothesis 3.4.} 1. $T_n \subseteq \omega^2$ is a tree with no maximal nodes (for $n < \omega$);

2. $B = \bigcup_{n<\omega} \text{lim}(T_n)$, $\bar{T} = \langle T_n : n < \omega \rangle$;

3. $2 \leq \iota < \omega$, $k = 2\iota$.

\textbf{Definition 3.5.} Let $M_{\bar{T},k}$ consist of all tuples 

\[m = (\ell_m, u_m, \bar{h}_m, \bar{g}_m) = (\ell, u, \bar{h}, \bar{g})\]

such that:
(a) \(0 < \ell < \omega, u \subseteq \ell 2\) and \(2 \leq |u|\);

(b) \(\bar{h} = \langle h_i : i < \iota \rangle, \bar{g} = \langle g_i : i < \iota \rangle\) and for each \(i < \iota\) we have
\[
h_i : u^{(2)} \rightarrow \omega \quad \text{and} \quad g_i : u^{(2)} \rightarrow \bigcup_{n<\omega} (T_n \cap \ell 2)
\]

(remember \(u^{(2)} = \{(\eta, \nu) \in u \times u : \eta \neq \nu\}\);

(c) \(g_i(\eta, \nu) \in T_{h_i(\eta, \nu)} \cap \ell 2\) for all \((\eta, \nu) \in u^{(2)}, i < \iota\);

(d) if \((\eta, \nu) \in u^{(2)}\) and \(i < \iota\), then \(\eta + g_i(\eta, \nu) = \nu + g_i(\nu, \eta)\);

(e) for any \((\eta, \nu) \in u^{(2)}\), there are no repetitions in the sequence \(\langle g_i(\eta, \nu), g_i(\nu, \eta) : i < \iota \rangle\).

**Definition 3.6.** Assume \(m = (\ell, u, \bar{h}, \bar{g}) \in M_{\ell,k}\) and \(\rho \in \ell 2\). We define \(m + \rho = (\ell', u', \bar{h}', \bar{g}')\) by

- \(\ell' = \ell, u' = \{\eta + \rho : \eta \in u\}\),
- \(\bar{h}' = \langle h_i' : i < \iota \rangle\) where \(h_i' : (u')^{(2)} \rightarrow \omega\) are such that \(h_i'(\eta+\rho, \nu+\rho) = h_i(\eta, \nu)\) for \((\eta, \nu) \in u^{(2)}\),
- \(\bar{g}' = \langle g_i' : i < \iota \rangle\) where \(g_i' : (u')^{(2)} \rightarrow \bigcup_{n<\omega} (T_n \cap \ell 2)\) are such that \(g_i'(\eta+\rho, \nu+\rho) = g_i(\eta, \nu)\) for \((\eta, \nu) \in u^{(2)}\).

Also if \(\rho \in \omega 2\), then we set \(m + \rho = m + (\rho|\ell)\).

**Observation 3.7.**

1. If \(m \in M_{\ell,k}\) and \(\rho \in \ell m 2\), then \(m + \rho \in M_{\ell,k}\).

2. For each \(\rho \in \omega 2\) the mapping
\[
M_{\ell,k} \rightarrow M_{\ell,k} : m \mapsto m + \rho
\]
is a bijection.

**Definition 3.8.** Assume \(m, n \in M_{\ell,k}\). We say that \(n\) extends \(m\) \((m \sqsubseteq n\) in short) if and only if:

- \(\ell_m \leq \ell_n, u_m = \{\eta|\ell_m : \eta \in u_n\}\), and
• for every \((\eta, \nu) \in (u_\mathfrak{m})^2\) such that \(\eta|\ell_\mathfrak{m} \neq \nu|\ell_\mathfrak{m}\) and each \(i < \iota\) we have
\[
h_i^\mathfrak{m}(\eta|\ell_\mathfrak{m}, \nu|\ell_\mathfrak{m}) = h_i^\mathfrak{n}(\eta, \nu) \quad \text{and} \quad g_i^\mathfrak{m}(\eta|\ell_\mathfrak{m}, \nu|\ell_\mathfrak{m}) = g_i^\mathfrak{n}(\eta, \nu)|\ell_\mathfrak{m}.
\]

**Definition 3.9.** We define a function\(^3\) \(\text{ndrk} : M_{\bar{T}, k} \to \text{ON} \cup \{\infty\}\) declaring inductively when \(\text{ndrk}(\mathfrak{m}) \geq \alpha\) (for an ordinal \(\alpha\)).

• \(\text{ndrk}(\mathfrak{m}) \geq 0\) always;

• if \(\alpha\) is a limit ordinal, then
\[
\text{ndrk}(\mathfrak{m}) \geq \alpha \iff (\forall \beta < \alpha)(\text{ndrk}(\mathfrak{m}) \geq \beta);
\]

• if \(\alpha = \beta + 1\), then \(\text{ndrk}(\mathfrak{m}) \geq \alpha\) if and only if for every \(\nu \in u_\mathfrak{m}\) there is \(\mathfrak{n} \in M_{\bar{T}, k}\) such that \(\ell_\mathfrak{n} > \ell_\mathfrak{m}\), \(\mathfrak{m} \sqsubseteq \mathfrak{n}\) and \(\text{ndrk}(\mathfrak{n}) \geq \beta\) and
\[
|\{\eta \in u_\mathfrak{n} : \nu \triangleleft \eta\}| \geq 2;
\]

• \(\text{ndrk}(\mathfrak{m}) = \infty\) if and only if \(\text{ndrk}(\mathfrak{m}) \geq \alpha\) for all ordinals \(\alpha\).

We also define
\[
\text{NDRK}(\bar{T}) = \sup\{\text{ndrk}(\mathfrak{m}) + 1 : \mathfrak{m} \in M_{\bar{T}, k}\}.
\]

**Lemma 3.10.** 1. The relation \(\sqsubseteq\) is a partial order on \(M_{\bar{T}, k}\).

2. If \(\mathfrak{m}, \mathfrak{n} \in M_{\bar{T}, k}\) and \(\mathfrak{m} \sqsubseteq \mathfrak{n}\) and \(\alpha \leq \text{ndrk}(\mathfrak{n})\), then \(\alpha \leq \text{ndrk}(\mathfrak{m})\).

3. The function \(\text{ndrk}\) is well defined.

4. If \(\mathfrak{m} \in M_{\bar{T}, k}\) and \(\rho \in \omega^2\) then \(\text{ndrk}(\mathfrak{m}) = \text{ndrk}(\mathfrak{m} + \rho)\).

5. If \(\mathfrak{m} \in M_{\bar{T}, k}\), \(\nu \in u_\mathfrak{m}\) and \(\text{ndrk}(\mathfrak{m}) \geq \omega_1\), then there is an \(\mathfrak{n} \in M_{\bar{T}, k}\) such that \(\mathfrak{m} \sqsubseteq \mathfrak{n}\), \(\text{ndrk}(\mathfrak{n}) \geq \omega_1\), and
\[
|\{\eta \in u_\mathfrak{n} : \nu \triangleleft \eta\}| \geq 2.
\]

6. If \(\mathfrak{m} \in M_{\bar{T}, k}\) and \(\infty > \text{ndrk}(\mathfrak{m}) = \beta > \alpha\), then there is \(\mathfrak{n} \in M_{\bar{T}, k}\) such that \(\mathfrak{m} \sqsubseteq \mathfrak{n}\) and \(\text{ndrk}(\mathfrak{n}) = \alpha\).

\(^3\) \text{ndrk} stands for \text{nondisjointness rank.}
7. If \( \text{NDRK}(T) \geq \omega_1 \), then \( \text{NDRK}(T) = \infty \).

8. Assume \( m \in M_{T,k} \) and \( u' \subseteq u_m, |u'| \geq 2 \). Put \( \ell' = \ell_m, h'_i = h_i^m |u^{(2)} \) and \( g'_i = g_i^m |u^{(2)} \) (for \( i < \iota \)), and let \( m|u' = (\ell', u', h', g') \). Then \( m|u' \in M_{T,k} \) and \( \text{nrdk}(m) \leq \text{nrdk}(m|u') \).

**Proof.** (1) Straightforward.

(2) Induction on \( \alpha \). If \( \alpha = \alpha_0 + 1 \) and \( n' \sqsubset n \) is one of the witnesses used to claim that \( \text{nrdk}(n) \geq \alpha_0 + 1 \), then this \( n' \) can also be used for \( m \). Hence we can argue the successor step of the induction. The limit steps are even easier.

(3) One has to show that if \( \beta < \alpha \) and \( \text{nrdk}(m) \geq \alpha \), then \( \text{nrdk}(m) \geq \beta \).

This can be shown by induction on \( \alpha \): at the successor stage if \( n \) is one of the witnesses used to claim that \( \text{nrdk}(m) \geq \alpha + 1 \), then \( \text{nrdk}(n) \geq \alpha \). By (2) we get \( \text{nrdk}(m) \geq \alpha \) and by the inductive hypothesis \( \text{nrdk}(m) \geq \gamma \) for \( \gamma \leq \alpha \). Limit stages are easy too.

(4) Clear.

(5) Let \( \mathcal{N} \) be the collection of all \( n \in M_{T,k} \) such that \( m \subseteq n \) and \( |\{ \eta \in u_n : \nu < \eta \}| \geq 2 \). If \( \text{nrdk}(n_0) \geq \omega_1 \) for some \( n_0 \in \mathcal{N} \), then we are done. So suppose towards contradiction that there is no such \( n_0 \). Then, as \( \mathcal{N} \) is countable,
\[
\alpha_0 \overset{\text{def}}{=} \sup \{ \text{nrdk}(n) + 1 : n \in \mathcal{N} \} < \omega_1.
\]
But \( \text{nrdk}(m) \geq \alpha_0 + 1 \) implies that \( \text{nrdk}(n_1) \geq \alpha_0 \) for some \( n_1 \in \mathcal{N} \), a contradiction.

(6) Induction on ordinals \( \beta \) (for all \( \alpha < \beta \)). The main point is that if \( \text{nrdk}(m) = \beta \), then for some \( \nu \in u_m \) we cannot find \( n \) as needed for witnessing \( \text{nrdk}(m) \geq \beta + 1 \), but for each \( \gamma < \beta \) we can find \( n \) needed for \( \text{nrdk}(m) \geq \gamma + 1 \). Therefore for each \( \gamma < \beta \) we may find \( n \equiv m \) such that \( \gamma \leq \text{nrdk}(n) < \beta \).

(7) Follows from (6) above.

(8) Clearly \( (\ell', u', h', g') \in M_{T,k} \). By a straightforward induction on \( \alpha \) for all \( m \) and restrictions \( m|u' \), one shows that
\[
\alpha \leq \text{nrdk}(m) \Rightarrow \alpha \leq \text{nrdk}(m|u').
\]
\( \square \)
Proposition 3.11. The following conditions are equivalent.  

(a) $\text{NDRK}(\bar{T}) \geq \omega_1$.  

(b) $\text{NDRK}(\bar{T}) = \infty$.  

(c) There is a perfect set $P \subseteq \omega_2$ such that  
   $$(\forall \eta, \nu \in P)\left((|B + \eta) \cap (B + \nu)| \geq k\right).$$  

(d) In some ccc forcing extension, there is $A \subseteq \omega_2$ of cardinality $\lambda_{\omega_1}$ such that  
   $$(\forall \eta, \nu \in A)\left((|B + \eta) \cap (B + \nu)| \geq k\right).$$  

Proof. (a) $\Rightarrow$ (b) This is Lemma 3.10(7).  

(b) $\Rightarrow$ (c) If $\text{NDRK}(\bar{T}) = \infty$ then there is $m_0 \in M_{\bar{T}, k}$ with $\text{ndrk}(m_0) \geq \omega_1$. Using Lemma 3.10(5) we may now choose a sequence $<m_j : j < \omega> \subseteq M_{\bar{T}, k}$ such that for each $j < \omega$:  

(i) $m_j \subseteq m_{j+1}$,  

(ii) $\text{ndrk}(m_j) \geq \omega_1$,  

(iii) $|\{\eta \in u_{m_{j+1}} : \nu < \eta\} | \geq 2$ for each $\nu \in u_{m_j}$.  

Let $P = \{\rho \in \omega^2 : (\forall j < \omega)(\rho|\ell_{m_j} \in u_{m_j})\}$. Clearly, $P$ is a perfect set. For $\eta, \nu \in P, \eta \neq \nu$, let $j_0$ be the smallest such that $\eta|\ell_m \neq \nu|\ell_m$ and let  

$$G_i(\eta, \nu) = \bigcup \{g^{m_j}_i(\eta|\ell_{m_j}, \nu|\ell_{m_j}) : j \geq j_0\} \in \lim\left(T_{h_i^{m_{j_0}}}(\eta|\ell_{m_{j_0}}, \nu|\ell_{m_{j_0}})\right)$$  

for $i < \iota$. Then $G_i : P^{(2)} \rightarrow B$ and for $(\eta, \nu) \in P^{(2)}$ and $i < \iota$:  

$$\eta + G_i(\eta, \nu) = \nu + G_i(\nu, \eta) \quad \text{and} \quad \eta + G_i(\nu, \eta) = \nu + G_i(\eta, \nu).$$  

Moreover, there are no repetitions in the sequence $<G_i(\eta, \nu), G_i(\nu, \eta) : i < \iota>$. Hence, for distinct $\eta, \nu \in P$ we have $|(B + \eta) \cap (B + \nu)| \geq 2\iota = k$.  

(c) $\Rightarrow$ (d) Assume (c). Let $\kappa = \beth_{\omega_1}$. By Corollary 2.11 we know that $\models_{\kappa} \lambda_{\omega_1} \leq c$. Remembering Proposition 3.3(1,2), we note that the formula “$P \times P \subseteq \text{std}_k(B)$” is $\Pi^1_1$, so it holds in the forcing extension by $\mathbb{C}_\kappa$. Now we easily conclude (d).
(d) ⇒ (a) Assume (d) and let \( \mathbb{P} \) be the ccc forcing notion witnessing this assumption, \( G \subseteq \mathbb{P} \) be generic over \( \mathbb{V} \). Let us work in \( \mathbb{V}[G] \).

Let \( \langle \eta_\alpha : \alpha < \lambda_{\omega_1} \rangle \) be a sequence of distinct elements of \( \omega \cdot 2 \) such that

\[
(\forall \alpha < \beta < \lambda_{\omega_1})(||B + \eta_\alpha|| \cap (B + \eta_\beta)| \geq k).
\]

Let \( \tau = \{ R_m : m \in M_{\tilde{T},k} \} \) be a (countable) vocabulary where each \( R_m \) is a \( |u_m| \)-ary relational symbol. Let \( \mathbb{M} = (\lambda_{\omega_1}, \{ R_m^\mathbb{M} \}_{m \in M_{\tilde{T},k}}) \) be the model in the vocabulary \( \tau \), where for \( m = (\ell, u, \bar{h}, \bar{g}) \in M_{\tilde{T},k} \) the relation \( R_m^\mathbb{M} \) is defined by

\[
R_m^\mathbb{M} = \big\{ (\alpha_0, \ldots, \alpha_{|u|-1}) \in (\lambda_{\omega_1})^{|u|} : \eta_{\alpha_0}[\ell], \ldots, \eta_{|u|-1}[\ell] = u \text{ and } \eta_{\alpha_j} + G_i(\alpha_j, \alpha_{j+1}) = \eta_{\alpha_{j+1}} \big\}.
\]

**Claim 3.11.1.**

1. If \( \alpha_0, \alpha_1, \ldots, \alpha_{j-1} < \lambda_{\omega_1} \) are distinct, \( j \geq 2 \), then for sufficiently large \( \ell < \omega \) there is \( m \in M_{\tilde{T},k} \) such that

\[
\ell_m = \ell, \quad u_m = \{ \eta_{\alpha_0}[\ell], \ldots, \eta_{\alpha_{j-1}}[\ell] \} \quad \text{and} \quad \mathbb{M} \models R_m[\alpha_0, \ldots, \alpha_{j-1}].
\]

2. Assume that \( m \in M_{\tilde{T},k} \), \( j < |u_m| \), \( \alpha_0, \alpha_1, \ldots, \alpha_{|u_m|-1} < \lambda_{\omega_1} \) and \( \alpha^* < \lambda_{\omega_1} \) are all pairwise distinct and such that

\[
\mathbb{M} \models R_m[\alpha_0, \ldots, \alpha_j, \ldots, \alpha_{|u_m|-1}]
\]

and

\[
\mathbb{M} \models R_m[\alpha_0, \ldots, \alpha_{j-1}, \alpha^*, \alpha_{j+1}, \ldots, \alpha_{|u_m|-1}].
\]

Then for every sufficiently large \( \ell > \ell_m \) there is \( n \in M_{\tilde{T},k} \) such that \( m \subseteq n \) and

\[
\ell_n = \ell, \quad u_n = \{ \eta_{\alpha_0}[\ell], \ldots, \eta_{\alpha_{|u_m|-1}}[\ell], \eta_{\alpha^*}[\ell] \}
\]

and

\[
\mathbb{M} \models R_n[\alpha_0, \ldots, \alpha_{|u_m|-1}, \alpha^*].
\]

3. If \( m \in M_{\tilde{T},k} \) and \( \mathbb{M} \models R_m[\alpha_0, \ldots, \alpha_{|u_m|-1}] \), then

\[
\text{rdr}(\{ \alpha_0, \ldots, \alpha_{|u_m|-1} \}, \mathbb{M}) \leq \text{ndrk}(m).
\]
Proof of the Claim. (1) For distinct $j_1, j_2 < j$ let $G_i(\alpha_{j_1}, \alpha_{j_2}) \in B$ (for $i < \iota$) be such that

$$\eta_{\alpha_{j_1}} + G_i(\alpha_{j_1}, \alpha_{j_2}) = \eta_{\alpha_{j_2}} + G_i(\alpha_{j_2}, \alpha_{j_1})$$

and there are no repetitions in the sequence $(G_i(\alpha_{j_1}, \alpha_{j_2}), G_i(\alpha_{j_2}, \alpha_{j_1}) : i < \iota)$. (Remember, $x \in (B + \eta_{\alpha_{j_1}}) \cap (B + \eta_{\alpha_{j_2}})$ if and only if $x + (\eta_{\alpha_{j_1}} + \eta_{\alpha_{j_2}}) \in (B + \eta_{\alpha_{j_1}}) \cap (B + \eta_{\alpha_{j_2}})$, so the choice of $G_i(\alpha_{j_1}, \alpha_{j_2})$ is possible by the assumptions on $\eta_{\alpha}$'s.) Suppose that $\ell < \omega$ is such that for any distinct $j_1, j_2 < j$ we have $\eta_{\alpha_{j_1}}|\ell \neq \eta_{\alpha_{j_2}}|\ell$ and there are no repetitions in the sequence $(G_i(\alpha_{j_1}, \alpha_{j_2})|\ell, G_i(\alpha_{j_2}, \alpha_{j_1})|\ell : i < \iota)$. Now let $u = \{\eta_{\alpha_j}|\ell : j' < j\}$, and for $i < \iota$ let $g_i(\eta_{\alpha_j}|\ell, \eta_{\alpha_{j_2}}|\ell) = G_i(\alpha_{j_1}, \alpha_{j_2})|\ell$, and let $h_i(\eta_{\alpha_j}|\ell, \eta_{\alpha_{j_2}}|\ell) < \omega$ be such that $G_i(\alpha_{j_1}, \alpha_{j_2}) \in \lim \{T_i(\eta_{\alpha_j}|\ell, \eta_{\alpha_{j_2}}|\ell)\}$. This defines $m = (\ell, u, \bar{h}, \bar{g}) \in M_{T, k}$ and easily $M \models R_m[\alpha_0, \ldots, \alpha_{j-1}]$.

(2) An obvious modification of the argument above.

(3) By induction on $\beta$ we show that for every $m \in M_{T, k}$ and all $\alpha_0, \ldots, \alpha_{|u_m|-1} < \lambda_{\omega_1}$ such that $M \models R_m[\alpha_0, \ldots, \alpha_{|u_m|-1}]$:

$$\beta \leq \text{rk}(\{\alpha_0, \ldots, \alpha_{|u_m|-1}\}, M)$$

implies $\beta \leq \text{nrdr}(m)$.

Steps $\beta = 0$ and $\beta$ is limit:

**Step $\beta = \gamma + 1$:** Suppose $m \in M_{T, k}$ and $\alpha_0, \ldots, \alpha_{|u_m|-1} < \lambda_{\omega_1}$ are such that $M \models R_m[\alpha_0, \ldots, \alpha_{|u_m|-1}]$ and $\gamma + 1 \leq \text{rk}(\{\alpha_0, \ldots, \alpha_{|u_m|-1}\}, M)$. Let $\nu \in u_m$, so $\nu = \eta_{\alpha_j}|\ell_m$ for some $j < |u_m|$. Since

$$\gamma + 1 \leq \text{rk}(\{\alpha_0, \ldots, \alpha_{|u_m|-1}\}, M)$$

we may find $\alpha^* \in \lambda_{\omega_1} \setminus \{\alpha_0, \ldots, \alpha_{|u_m|-1}\}$ such that

$$M \models R_m[\alpha_0, \ldots, \alpha_{j-1}, \alpha^*, \alpha_{j+1}, \ldots, \alpha_{|u|-1}]$$

and $\text{rk}(\{\alpha_0, \ldots, \alpha_{|u|-1}, \alpha^*\}, M) \geq \gamma$. Taking sufficiently large $\ell$ we may use clause (2) to find $n \in M_{T, k}$ such that $m \subseteq n$, $\ell_n = \ell$ and $M \models R_n[\alpha_0, \ldots, \alpha_{|u_m|-1}, \alpha^*]$ and $|\{\eta \in u_n : \nu < \eta\}| \geq 2$. By the inductive hypothesis we have also $\gamma \leq \text{nrdr}(n)$. Now we may easily conclude that $\gamma + 1 \leq \text{nrdr}(m)$. \qed
By the definition of $\lambda_{\omega_1}$,
\[(\circ) \sup \{\text{rk}(w, M) : \emptyset \neq w \in [\lambda_{\omega_1}]^{<\omega}\} \geq \omega_1\]

Now, suppose that $\beta < \omega_1$. By $(\circ)$, there are distinct $\alpha_0, \ldots, \alpha_{j-1} < \lambda_{\omega_1}$, $j \geq 2$, such that $\text{rk}(\{\alpha_0, \ldots, \alpha_{j-1}\}, M) \geq \beta$. By Claim 3.11.1(1) we may find $m \in M^\omega_{\bar{T}}$ such that $M \models R_m[\alpha_0, \ldots, \alpha_{j-1}]$. Then by Claim 3.11.1(3) we also have $\text{ndrk}(m) \geq \beta$. Consequently, $\text{NDRK}(\bar{T}) \geq \omega_1$.

All the considerations above where carried out in $V[G]$. However, the rank function $\text{ndrk}$ is absolute, so we may also claim that in $V$ we have $\text{NDRK}(\bar{T}) \geq \omega_1$. \hfill \Box

**Corollary 3.12.** Assume that $\varepsilon \leq \omega_1$ and $\text{Pr}_\varepsilon(\lambda)$. If there is $A \subseteq \omega^2$ of cardinality $\lambda$ such that
\[\forall \eta, \nu \in A \left( |(B + \eta) \cap (B + \nu)| \geq k \right),\]
then $\text{NDRK}(\bar{T}) \geq \varepsilon$.

**Proof.** This is essentially shown by the proof of the implication $(d) \Rightarrow (a)$ of Proposition 3.11. \hfill \Box

4. The forcing

In this section we construct a forcing notion adding a sequence $\bar{T}$ of sub-trees of $\omega^>2$ such that $\text{NDRK}(\bar{T}) < \omega_1$. The sequence $\bar{T}$ will be added by finite approximations, so it will be convenient to have finite version of Definition 3.5.

**Definition 4.1.** Assume that
- $2 \leq \iota < \omega$, $k = 2\iota$, and $0 < n, M < \omega$,
- $\bar{t} = \langle t_m : m < M \rangle$, and each $t_m$ is a sub-tree of $n^\geq 2$ in which all terminal branches are of length $n$,
- $T_j \subseteq \omega^>2$ (for $j < \omega$) are trees with no maximal nodes, $\bar{T} = \langle T_j : j < \omega \rangle$ and $t_m = T_m \cap n^\geq 2$ for $m < M$,
- $M^\omega_{\bar{T}, k}$ is defined as in Definition 3.5.
1. Let \( M_{i,k}^n \) consist of all tuples \( m = (\ell, u, h, g) \in M_{T,k} \) such that \( \ell \leq n \) and \( \text{rng}(h^m) \subseteq M \) for each \( i < \ell \).

2. Assume \( m, n \in M_{i,k}^n \). We say that \( m, n \) are essentially the same \( (m \equiv* n \text{ in short}) \) if and only if:

- \( \ell = \ell, u = u \) and
- for each \( (\eta, \nu) \in (u, u)^{(2)} \) we have
  \[
  \{ \{g_i^m(\eta, \nu), g_i^m(\nu, \eta)\} : i < \ell \} = \{ \{g_i^n(\eta, \nu), g_i^n(\nu, \eta)\} : i < \ell \},
  \]
  and for \( i, j < \ell \):
  - if \( g_i^m(\eta, \nu) = g_j^m(\eta, \nu) \), then \( h_i^m(\eta, \nu) = h_j^m(\eta, \nu) \).
  - if \( g_i^m(\eta, \nu) = g_j^m(\nu, \eta) \), then \( h_i^m(\eta, \nu) = h_j^m(\nu, \eta) \).

3. Assume \( m, n \in M_{i,k}^n \). We say that \( n \) essentially extends \( m \) \( (m \sqsubset^* n \text{ in short}) \) if and only if:

- \( \ell \leq \ell, u = \{\eta|\ell : \eta \in u\} \), and
- for each \( (\eta, \nu) \in (u, u)^{(2)} \) such that \( \eta|\ell \neq \nu|\ell \) we have
  \[
  \{ \{g_i^m(\eta|\ell, \nu|\ell), g_i^m(\nu|\ell, \eta|\ell)\} : i < \ell \} \]
  \[
  = \{ \{g_i^n(\eta, \nu)|\ell, g_i^n(\nu, \eta)|\ell\} : i < \ell \},
  \]
  and for \( i, j < \ell \):
  - if \( g_i^m(\eta|\ell, \nu|\ell) = g_j^m(\eta, \nu)|\ell \), then \( h_i^m(\eta|\ell, \nu|\ell) = h_j^m(\eta, \nu) \).
  - if \( g_i^m(\eta|\ell, \nu|\ell) = g_j^m(\nu, \eta)|\ell \), then \( h_i^m(\eta|\ell, \nu|\ell) = h_j^m(\nu, \eta) \).

Observation 4.2. If \( m \in M_{i,k}^n \) and \( \rho \in \ell_m 2 \), then \( m + \rho \in M_{i,k}^n \) (remember Definition 3.6).

Lemma 4.3. Let \( 0 < \ell < \omega \) and let \( B \subseteq \ell^2 \) be a linearly independent set of vectors \( (in \ell^2, +) \text{ over } (2, +2, \cdot2)) \).

1. If \( A \subseteq \ell^2, |A| \geq 5 \) and \( A + A \subseteq B + B \), then for a unique \( x \in \ell^2 \) we have \( A + x \subseteq B \).

2. Let \( b^* \in B \). Suppose that \( \rho_i^0, \rho_i^1 \in (B \cup (b^* + B)) \setminus \{0, b^*\} \) \( (for i < 3) \) are such that
   (a) there are no repetitions in \( \{\rho_i^0, \rho_i^1 : i < 3\} \), and
\( \rho_i^0 + \rho_i^1 = \rho_j^0 + \rho_j^1 \) for \( i < j < 3 \).

Then \( \{\rho_i^0, \rho_i^1\} : i < 3 \subseteq \{b, b + b^*\} : b \in B, \ b \neq b^*\} \).

**Proof.** Easy, for (1) see e.g. [5, Lemma 2.3]. \( \square \)

**Theorem 4.4.** Assume \( \text{NPr}_{\omega_1}(\lambda) \) and let \( 3 \leq i < \omega \). Then there is a ccc forcing notion \( \mathbb{P} \) of size \( \lambda \) such that

\( \mathbb{P} \models \text{"for some } \Sigma_0^0 \text{n-pots-set } B \subseteq \omega_2 \text{ there is a sequence } \langle \eta_\alpha : \alpha < \lambda \rangle \text{ of distinct elements of } \omega_2 \text{ such that } \left| (\eta_\alpha + B) \cap (\eta_\beta + B) \right| \geq 2^\iota \text{ for all } \alpha, \beta < \lambda. \)

**Proof.** If \( Q \subseteq \omega_2 \) is a countable infinite subgroup of \( \omega_2 \) then \( Q \) is n-pots but \( Q \) has \( \omega \)-many pairwise \( \omega \)-nondisjoint translations. So we may assume that \( \lambda \) is uncountable.

Fix a countable vocabulary \( \tau = \{R_{n, \zeta} : n, \zeta < \omega\} \), where \( R_{n, \zeta} \) is an \( n \)-ary relational symbol (for \( n, \zeta < \omega \)). By the assumption on \( \lambda \), we may fix a model \( \mathbb{M} = (\lambda, \{R_{n, \zeta}^\mathbb{M}\}_{n, \zeta < \omega}) \) in the vocabulary \( \tau \) with the universe \( \lambda \) and an ordinal \( \alpha^* < \omega_1 \) such that:

\( (\otimes)_a \) for every \( n \) and a quantifier free formula \( \varphi(x_0, \ldots, x_{n-1}) \in \mathcal{L}(\tau) \) there is \( \zeta < \omega \) such that for all \( a_0, \ldots, a_{n-1} \in \lambda \),

\[ \mathbb{M} \models \varphi[a_0, \ldots, a_{n-1}] \Leftrightarrow R_{n, \zeta}[a_0, \ldots, a_{n-1}], \]

\( (\otimes)_b \sup \{\text{rk}(v, \mathbb{M}) : \emptyset \neq v \in [\lambda]^{<\omega}\} < \alpha^* \),

\( (\otimes)_c \) the rank of every singleton is at least 0.

For a nonempty finite set \( v \subseteq \lambda \) let \( \text{rk}(v) = \text{rk}(v, \mathbb{M}) \), and let \( \zeta(v) < \omega \) and \( k(v) < |v| \) be such that \( R_{|v|, \zeta(v)}^v, k(v) \) witness the rank of \( v \). Thus letting \( \{a_0, \ldots, a_k, \ldots a_{n-1}\} \) be the increasing enumeration of \( v \) and \( k = k(v) \) and \( \zeta = \zeta(v) \), we have

\( (\otimes)_d \) if \( \text{rk}(v) \geq 0 \), then \( \mathbb{M} \models R_{n, \zeta}[a_0, \ldots, a_k, \ldots a_{n-1}] \) but there is no \( a \in \lambda \setminus v \) such that

\( \text{rk}(v \cup \{a\}) \geq \text{rk}(v) \) and \( \mathbb{M} \models R_{n, \zeta}[a_0, \ldots, a_{k-1}, a, a_{k+1}, \ldots, a_{n-1}] \).
If $\text{rk}(v) = -1$, then $\mathbb{M} \models R_{n, \zeta}[a_0, \ldots, a_k, \ldots, a_{n-1}]$ but the set
\[
\{ a \in \lambda : \mathbb{M} \models R_{n, \zeta}[a_0, \ldots, a_{k-1}, a, a_{k+1}, \ldots, a_{n-1}] \}
\]
is countable.

Without loss of generality we may also require that (for $\zeta = \zeta(v)$, $n = |v|$)
\[(\circ)_f \text{ for every } b_0, \ldots, b_{n-1} < \lambda \]
\[
\text{if } \mathbb{M} \models R_{n, \zeta}[b_0, \ldots, b_{n-1}] \text{ then } b_0 < \ldots < b_{n-1}.
\]

Now we will define a forcing notion $\mathbb{P}$. A condition $p$ in $\mathbb{P}$ is a tuple
\[
(w^p, n^p, M^p, \bar{\eta}^p, \bar{t}^p, \bar{r}^p, \bar{h}^p, \bar{g}^p, \mathcal{M}^p) = (w, n, M, \bar{\eta}, \bar{t}, \bar{r}, \bar{h}, \bar{g}, \mathcal{M})
\]
such that the following demands $(\ast)_1 - (\ast)_{11}$ are satisfied.

\[
(\ast)_1 \, w \in [\lambda]^{<\omega}, |w| \geq 5, 0 < n, M < \omega.
\]

\[
(\ast)_2 \, \bar{\eta} = \langle \eta_\alpha : \alpha \in w \rangle \text{ is a sequence of linearly independent vectors in } n^2 \text{ (over the field } \mathbb{Z}_2); \text{ so in particular } \eta_\alpha \in n^2 \text{ are pairwise distinct non-zero sequences (for } \alpha \in w).
\]

\[
(\ast)_3 \, \bar{t} = \langle t_m : m < M \rangle, \text{ where } \emptyset \neq t_m \subseteq n^2 \text{ for } m < M \text{ is a tree in which all terminal branches are of length } n \text{ and } t_m \cap t_{m'} \cap n^2 = \emptyset \text{ for } m < m' < M.
\]

\[
(\ast)_4 \, \bar{r} = \langle r_m : m < M \rangle, \text{ where } 0 < r_m \leq n \text{ for } m < M.
\]

\[
(\ast)_5 \, \bar{h} = \langle h_i : i < \iota \rangle, \text{ where } h_i : w^{(2)} \to M.
\]

\[
(\ast)_6 \, \bar{g} = \langle g_i : i < \iota \rangle, \text{ where } g_i : w^{(2)} \to \bigcup_{m < M} (t_m \cap n^2), \text{ and } g_i(\alpha, \beta) \in t_{h_i(\alpha, \beta)} \text{ and } \eta_\alpha + g_i(\alpha, \beta) = \eta_\beta + g_i(\beta, \alpha) \text{ for } (\alpha, \beta) \in w^{(2)} \text{ and } i < \iota.
\]

\[
(\ast)_7 \text{ There are no repetitions in the list } \langle g_i(\alpha, \beta) : i < \iota, (\alpha, \beta) \in w^{(2)} \rangle.
\]

\[
(\ast)_8 \, \mathcal{M} \text{ consists of all those } m \in \mathbb{M}_{\ell, k}^n \text{ (see Definition 4.1) that for some } \ell_*, w_* \text{ we have}
\]
To define the order $\leq$ of $\mathbb{P}$ we declare for $p, q \in \mathbb{P}$ that $p \leq q$ if and only if

- $w^p \subseteq w^q$, $n^p \leq n^q$, $M^p \leq M^q$, and
- $t^p_m = t^q_m \cap n^{p\geq2}$ and $r^p_m = r^q_m$ for all $m < M^p$, and
\[\eta^p_\alpha \leq \eta^q_\alpha \text{ for all } \alpha \in w^p, \text{ and}\]
\[h^q_i|(w^p)^{(2)} = h^p_i \text{ and } g^p_i(\alpha, \beta) \leq g^q_i(\alpha, \beta) \text{ for } i < \iota \text{ and } (\alpha, \beta) \in (w^p)^{(2)}.\]

**Claim 4.4.1.** Assume \(p = (w, n, M, \bar{h}, \bar{t}, \bar{h}, \bar{g}, \bar{M}) \in \mathbb{P}. \) If \(m \in M^p_{i,k} \) is such that \(\ell_m = n \) and \(|u_m| \geq 5, \) then for some \(\rho \in n^2 \) and \(n \in \mathcal{M} \) we have \((m + \rho) \not{\equiv} n.\)

**Proof of the Claim.** Let \(m \in M^p_{i,k} \) be such that \(\ell_m = n. \) It follows from Definition 3.5(d,e) and clauses \((*)_6 + (*)_{11} \) that

\[(\square) \text{ for every } (\nu, \eta) \in (u_m)^{(2)} \text{ there is } (\alpha, \beta) \in w^{(2)} \text{ such that } \nu + \eta = \eta_\alpha + \eta_\beta.\]

By Lemma 4.3 for some \(\rho \) we have \(u_m + \rho \subseteq \{\eta_\alpha : \alpha \in w\}. \) Let \(w_0 = \{\alpha \in w : \eta_\alpha + \rho \in u_m\} \) and \(n = m^p(n, w_0) \in \mathcal{M}. \) Using clauses \((*)_{11} \) and \((*)_6 \) we easily conclude \((m + \rho) \not{\equiv} n. \) (Note that since \(t_m \cap t_{m'} \cap n^2 = \emptyset \) for \(m < m' < M, \) \(h^m_i(\eta, \nu) \) is determined by \(g^m_i(\eta, \nu).\)) \(\square\)

**Claim 4.4.2.**

1. \(\mathbb{P} \neq \emptyset \) and \((\mathbb{P}, \leq) \) is a partial order.

2. For each \(\beta < \lambda \) and \(n_0, M_0 < \omega \) the set

\[D^{n_0,M_0}_{\beta} = \{p \in \mathbb{P} : n^p > n_0 \land M^p > M_0 \land \beta \in w^p\}\]

is open dense in \(\mathbb{P}.\)

**Proof of the Claim.** (1) Straightforward.

(2) Let \(p \in \mathbb{P}, \beta \in \lambda \setminus w^p. \) Put \(N = |w^p| \cdot \iota + 2. \)

We will define a condition \(q \in \mathbb{P} \) such that \(q \geq p \) and

\[w^q = w^p \cup \{\beta\}, \quad n^q = n^p + N > n^p + 1, \quad M^q = M^p + N - 2 > M^p + 1.\]

For \(\alpha \in w^p \) we set \(\eta^q_\alpha = \eta^p_\alpha \upharpoonright (0, \ldots, 0) \) and we also let

\[\eta^q_\beta = (0, \ldots, 0) \upharpoonright (1, \ldots, 1).\]

Next, if \((\alpha_0, \alpha_1) \in (w^p)^{(2)}, \) then for all \(i < \iota\)

\[h^q_i(\alpha_0, \alpha_1) = h^p_i(\alpha_0, \alpha_1) \quad \text{and} \quad g^q_i(\alpha_0, \alpha_1) = g^p_i(\alpha_0, \alpha_1) \upharpoonright (0, \ldots, 0).\]

If \(\alpha \in w^p \) and \(j = |w^p \cap \alpha|, \) then for \(i < \iota:\)
Demands \( (\ast \nexists \beta/\mu \gamma \cdot) \) and \( \beta \in w_0 \cup w_1 \)

Then letting \( \ell^* = \min(\ell, n^\beta) \) and \( \rho^* = \rho|\ell^* \) we see that \( \mathfrak{m}^\beta(\ell^*, w_0) \doteq \mathfrak{m}^\beta(\ell^*, w_1) + \rho^* \) (and both belong to \( \mathcal{M}^\beta \)). Hence clause \( (*)_9 \) for \( p \) applies.

**Case 2:** \( \beta \in w_0 \cup w_1 \)

Say, \( \beta \in w_0 \). If \( \alpha \in w_0 \setminus \{\beta\} \), then \( h_1^\gamma(\alpha, \beta) = h_1^\gamma(\beta, \alpha) \geq M^\beta \) and \( r_1^\gamma(\alpha, \beta) = n^\theta \). Consequently, \( \ell = n^\gamma \). Moreover,

\[
(\gamma, \delta) \in (w^\gamma)^{(2)} \land h_j^\gamma(\gamma, \delta) = h_j^\gamma(\alpha, \beta) \implies \{\gamma, \delta\} = \{\alpha, \beta\}.
\]

Therefore, \( \beta \in w_1 \) and \( w_1 = w_0 \) and since \( |w_1| \geq 5 \), the linear independence of \( \eta \) implies \( \rho = 0 \).
RE (\(*)_{10}\) : Concerning clause (\*)_{10}, suppose that \(m^q(\ell_0, w_0), m^q(\ell_1, w_1) \in M^q, \alpha \in w_0, |\alpha \cap w_0| = k(w_0), \text{rk}(w_0) = -1, \) and \(m^q(\ell_0, w_0) \not\subset m^q(\ell_1, w_1).\) Assume towards contradiction that there are \(\alpha_0, \alpha_1 \in w_1\) such that

\[
\eta^q_{\alpha_0} \upharpoonright \ell_1 \neq \eta^q_{\alpha_1} \upharpoonright \ell_1 \land \eta^q_{\alpha_0} \upharpoonright \ell_0 < \eta^q_{\alpha_1} \upharpoonright \ell_0 < \eta^q_{\alpha_1}.
\]

Suppose \(\beta \in w_0 \cup w_1.\) Then looking at the function \(h^q_i\) in a manner similar to considerations for clause (\*)_{9} we get \(\beta \in w_0 \cap w_1.\) Let \(\beta' \in w_0 \setminus \{\beta\}.\) Then \(h^q_i(\beta, \beta') \geq M^p\) and hence \(r^q_{h^0(\beta, \beta')} = n^q = \ell_0 = \ell_1,\) contradicting our assumptions. Therefore \(\beta \notin w_0 \cup w_1.\) But then we immediately get contradiction with clause (\*)_{10} for \(p.\)

RE (\*)_{11} : Let us argue that (\*)_{11} is satisfied as well and for this suppose that \(\rho^0_i, \rho^1_i \in \bigcap_{m < M^q} (t_m \cap n^q 2) (\text{for } i < \iota)\) are such that

(a) there are no repetitions in \(\langle \rho^0_i, \rho^1_i : i < \iota \rangle,\) and

(b) \(\rho^0_i + \rho^1_i = \rho^0_j + \rho^1_j\) for \(i < j < \iota.\)

Clearly, if

(\(\circ\))_{1} all \(\rho^0_i, \rho^1_i\) are from \(\bigcup_{m < M^p} t_m,\)

then we may use the condition (\*)_{11} for \(p\) and conclude that for some \(\alpha_0, \alpha_1 \in w^p\) we have

\[
\{\{\rho^0_i, \rho^1_i\} : i < \iota\} = \{\{g_i(\alpha_0, \alpha_1), g_i(\alpha_1, \alpha_0)\} : i < \iota\}.
\]

Now note that if \(\rho_0, \rho_1, \rho_2, \rho_3 \in \bigcup_{m < M^q} (t_m \cap n^q 2), \rho_0 + \rho_1 = \rho_2 + \rho_3\) and \(\rho_0 \in \bigcup_{m < M^p} (t_m \cap n^q 2)\) but \(\rho_1 \notin \bigcup_{m < M^p} (t_m \cap n^q 2),\) then \(\{\rho_0, \rho_1\} = \{\rho_2, \rho_3\}.\) Hence easily, if (\(\circ\))_{1} fails we must have

(\(\circ\))_{2} \(\rho^0_i, \rho^1_i \in \bigcup_{m = M^p} (t_m \cap n^q 2)\) for \(i < \iota.\)

But then necessarily

\[
\{\{\rho^0_i[n^p, n^q], \rho^1_i[n^p, n^q]\} : i < \iota\} \subseteq \{\{g_i(\alpha, \beta)[n^p, n^q], g_i(\beta, \alpha)[n^p, n^q]\} : i < \iota, \alpha \in w^p\}.
\]
procedure we may find an uncountable set $A$ demands ($\bar{\eta}_\alpha$)

\[
\{\{\rho_i^0, \rho_i^1\} : i < \iota\} = \{\{g_i(\alpha, \beta), g_i(\beta, \alpha)\} : i < \iota\}.
\]

One easily verifies that the condition $q$ is stronger than $p$. \qed

**Claim 4.4.3.** The forcing notion $P$ has the Knaster property.

**Proof of the Claim.** Suppose that $\langle p_\xi : \xi < \omega_1 \rangle$ is a sequence of pairwise distinct conditions from $P$ and let

\[
p_\xi = (w_\xi, n_\xi, M_\xi, \bar{n}_\xi, \bar{t}_\xi, \bar{r}_\xi, \bar{h}_\xi, g_\xi, M_\xi)
\]

where $\bar{n}_\xi = \langle n_\alpha^\xi : \alpha \in w_\xi\rangle$, $\bar{t}_\xi = \langle t_m^\xi : m < M_\xi\rangle$, $\bar{r}_\xi = \langle r_m^\xi : m < M_\xi\rangle$, and $\bar{h}_\xi = \langle h_i^\xi : i < \iota\rangle$, $g_\xi = \langle g_i^\xi : i < \iota\rangle$. By a standard $\Delta$–system cleaning procedure we may find an uncountable set $A \subseteq \omega_1$ such that the following demands $(*)_{12} - (*)_{15}$ are satisfied.

$(*)_{12}$ \{\(w_\xi : \xi \in A\) forms a $\Delta$–system.

$(*)_{13}$ If $\xi, \varsigma \in A$, then $|w_\xi| = |w_\varsigma|$, $n_\xi = n_\varsigma$, $M_\xi = M_\varsigma$, and $t_m^\xi = t_m^\varsigma$ and $r_m^\xi = r_m^\varsigma$ (for $m < M_\xi$).

$(*)_{14}$ If $\xi < \varsigma$ are from $A$ and $\pi : w_\xi \longrightarrow w_\varsigma$ is the order isomorphism, then

(a) $\pi(\alpha) = \alpha$ for $\alpha \in w_\xi \cap w_\varsigma$,

(b) if $\emptyset \neq v \subseteq w_\xi$, then $rk(v) = rk(\pi[v])$, $\zeta(v) = \zeta(\pi[v])$ and $k(v) = k(\pi[v])$,

(c) $\eta_\alpha^\xi = \eta_{\pi(\alpha)}$ (for $\alpha \in w_\xi$),

(d) $g_i(\alpha, \beta) = g_i(\pi(\alpha), \pi(\beta))$ and $h_i(\alpha, \beta) = h_i(\pi(\alpha), \pi(\beta))$ for $(\alpha, \beta) \in (w_\xi)^{(2)}$ and $i < \iota$.

and

$(*)_{15}$ $M_\xi = M_\varsigma$ (this actually follows from the previous demands).

Following the pattern of Claim 4.4.2(2) we will argue that for distinct $\xi, \varsigma$ from $A$ the conditions $p_\xi, p_\varsigma$ are compatible. So let $\xi, \varsigma \in A$, $\xi < \varsigma$ and let $\pi : w_\xi \longrightarrow w_\varsigma$ be the order isomorphism. We will define $q =$
(w, n, M, \eta, \bar{\eta}, \bar{r}, \bar{r}, \bar{h}, \bar{g}, M) where \eta = \langle \eta_0 : \alpha \in w \rangle, \bar{\eta} = \langle t_m : m < M \rangle, \
\bar{r} = \langle r_m : m < M \rangle, and \bar{h} = \langle h_i : i < \iota \rangle, \bar{g} = \langle g_i : i < \iota \rangle. 

Let w_\xi \cap w_\zeta = \{\alpha_0, \ldots, \alpha_{k-1}\}, w_\xi \setminus w_\zeta = \{\beta_0, \ldots, \beta_{\ell-1}\} and w_\zeta \setminus w_\xi = 
\{\gamma_0, \ldots, \gamma_{\ell-1}\} be the increasing enumerations.

We set \(N_0 = \iota \cdot \ell(\ell + k) + \iota \cdot \frac{\ell(\ell - 1)}{2} + 1, N = N_0 + \ell + 1, and we define 
\((*)_{16} \ w = w_\xi \cup w_\zeta, n = n_\xi + N, and M = M_\xi + 1; \)
\((*)_{17} \ \eta_\alpha = \eta_\alpha^N(0, \ldots, 0) \) for \(\alpha \in w_\xi \) and we also let for \(c < \ell \)
\[\eta_{\gamma_c} = \eta_{\gamma_c}^N(0) \cap (1, 1) \cap (0, \ldots, 0) \cap (1, \ldots, 1).\]

Next we are going to define \(h_i(\alpha, \beta) \) and \(g_i(\alpha, \beta) \) for \((\alpha, \beta) \in w^{(2)} \). For \(d < N_0 \) let
\[\nu_d = \langle 0, \ldots, 0 \rangle \cap (1) \cap (0, \ldots, 0) \in N_{0, 2}, \text{ and } \nu_d^* = 1 + \nu_d \in N_{0, 2}\]
and note that \(\{\nu_d : d < N_0 - 1 \} \cup \{1\} \) are linearly independent in \(N_{0, 2} \). Fix a bijection
\[\Theta : (k \times \ell \times \ell \times \{0\}) \cup (\{(a, b) \in \ell^2 : a < b \} \times \ell \times \{1\}) \cup (\ell \times \ell \times \ell \times \{2\}) \rightarrow N_0 - 1\]
and define \(h_i, g_i \) as follows.

\((*)_{18}^a \) If \((\alpha, \beta) \in (w_\xi)^{(2)} \) and \(i < \iota \), then
\[h_i(\alpha, \beta) = h_i^\xi(\alpha, \beta) \) and \(g_i(\alpha, \beta) = g_i^\xi(\alpha, \beta)^N(0, \ldots, 0).\]

\((*)_{18}^b \) If \(a < k, c < \ell \) and \(i < \iota \), then \(h_i(\alpha_a, \gamma_c) = h_i^\xi(\alpha_a, \gamma_c) \) and \(h_i(\gamma_c, \alpha_a) = h_i^\xi(\gamma_c, \alpha_a) \), and
\[g_i(\alpha_a, \gamma_c) = g_i^\xi(\alpha_a, \gamma_c)^\ell(1) \cap (1) \cap (0, \ldots, 0) \) for \(\Theta(a, c, i, 0) = (\ell, 1, \ldots, 1, \ell) \) and
\[g_i(\gamma_c, \alpha_a) = g_i^\xi(\gamma_c, \alpha_a)^\ell(1) \cap (1) \cap (0, \ldots, 0) \cap (1, \ldots, 1).\]
If \( b < c < \ell \) and \( i < \iota \), then \( h_i(\gamma_b, \gamma_c) = h_i^\xi(\gamma_b, \gamma_c) \), \( h_i(\gamma_c, \gamma_b) = h_i^\xi(\gamma_c, \gamma_b) \), and
\[
g_i(\gamma_b, \gamma_c) = g_i^\xi(\gamma_b, \gamma_c)^-1(1)^\sim(0, \ldots, 0)^\sim(1, \ldots, 1)^- = b \quad \text{and} \quad g_i(\gamma_c, \gamma_b) = g_i^\xi(\gamma_c, \gamma_b)^-1(1)^\sim(0, \ldots, 0)^\sim(1, \ldots, 1)^- = c
\]
(note: \( \nu_\Theta \) not \( \nu_\Theta^* \)).

If \( b < \ell \), \( c < \ell \) and \( b \neq c \) and \( i < \iota \), then \( h_i(\beta_b, \gamma_c) = h_i(\gamma_c, \beta_b) = M_\xi = M_\zeta \), and
\[
g_i(\beta_b, \gamma_c) = g_i^\xi(\beta_b, \beta_c)^-1(1)^\sim(0, \ldots, 0)^\sim(1, \ldots, 1)^- = c \quad \text{and} \quad g_i(\gamma_c, \beta_b) = g_i^\xi(\gamma_c, \gamma_b)^-1(1)^\sim(0, \ldots, 0)^\sim(1, \ldots, 1)^- = c
\]

If \( b < \ell \) and \( i < \iota \), then \( h_i(\beta_b, \gamma_b) = h_i(\gamma_b, \beta_b) = M_\xi = M_\zeta \), and
\[
g_i(\beta_b, \gamma_b) = \eta^\xi_{\beta_b}^- (1)^\sim(0, \ldots, 0)^\sim(1, \ldots, 1)^- = b \quad \text{and} \quad g_i(\gamma_b, \beta_b) = \eta^\xi_{\gamma_b}^- (1)^\sim(0, \ldots, 0)^\sim(1, \ldots, 1)^- = \ell
\]

We also set:

\[ (*)_{19} \quad r_m = r_m^\xi \quad \text{for } m < M_\xi, \quad r_{M_\xi} = n \quad \text{and if } m < M_\xi, \text{ then} \]
\[
t_m = \{ \eta \in n^{\geq 2} : \eta[\eta_{\xi} \in t_m^\xi \cap (\forall j < n)(n \leq j < |\eta| \Rightarrow \eta(j) = 0) \} \cup \{ g_i(\delta, \varepsilon)|n' : (\delta, \varepsilon) \in w^{(2)}, i < \iota, \text{ and } n' \leq n \text{ and } h_i(\delta, \varepsilon) = m \}
\]
and
\[
t_{M_\xi} = \{ g_i(\delta, \varepsilon)|n' : (\delta, \varepsilon) \in w^{(2)}, i < \iota, \text{ and } n' \leq n \text{ and } h_i(\delta, \varepsilon) = M_\xi \}.
\]

Now letting \( \mathcal{M} \) be defined by \( (*)_8 \) we claim that
\[
q = (w, n, M, \bar{n}, \bar{t}, \bar{r}, \bar{h}, \bar{g}, \mathcal{M}) \in \mathbb{P}.
\]

Demands \( (*)_1 - (*)_8 \) are pretty straightforward.
RE (⑨) : To justify clause (⑨)9, suppose that \( m(\ell, w'), m(\ell, w'') \in \mathcal{M}, \) \( \rho \in \ell \) and \( m(\ell, w') \div m(\ell, w'') + \rho, \) and consider the following three cases.

CASE 1: \( w' \subseteq w_\xi \)
Then for each \((\delta, \varepsilon) \in (w')^{(2)}\) we have \( h_i(\delta, \varepsilon) < M_\xi, \) so this also holds for \((\delta, \varepsilon) \in (w'')^{(2)}\). Consequently, either \( w'' \subseteq w_\xi \) or \( w'' \subseteq w_\zeta. \)

If \( w'' \subseteq w_\xi, \) then let \( \ell = \min(\ell, n_\xi) \) and consider \( m^{p_\xi}(w', \ell'), m^{p_\xi}(w'', \ell') \in \mathcal{M}_{\xi}. \) Using clause (⑨)9 for \( p_\xi \) we immediately obtain the desired conclusion.

If \( w'' \subseteq w_\zeta, \) then we let \( \ell = \min(\ell, n_\xi) \) and we consider \( m^{p_\xi}(w', \ell') \) and \( m^{p_\xi}(\pi^{-1}[w''], \ell') \) (both from \( \mathcal{M}_{\xi}. \) By (⑨)14, clause (⑨)9 for \( p_\xi \) applies to them and we get

- \( \text{rk}(w') = \text{rk}(\pi^{-1}[w'']), \) \( \zeta(w') = \zeta(\pi^{-1}[w'']), \) \( k(w') = k(\pi^{-1}[w'']) \)

- if \( \delta \in w', \varepsilon \in \pi^{-1}[w''] \) are such that \( |\delta \cap w'| = k(w') = k(\pi^{-1}[w'']) = |\varepsilon \cap \pi^{-1}[w'']|, \) then \( \eta^p_{\delta} |\ell'| + \rho = \eta^p_{\varepsilon} |\ell'. \)

By (⑨)14 this immediately implies the desired conclusion.

CASE 2: \( w' \subseteq w_\zeta \)
Same as the previous case, just interchanging \( \xi \) and \( \zeta. \)

CASE 3: \( w' \setminus w_\xi \neq \emptyset \neq w' \setminus w_\zeta \)
Then for some \((\delta, \varepsilon) \in (w')^{(2)}\) we have \( h_i(\delta, \varepsilon) = M_\zeta, \) so necessarily \( \ell = r_{M_\zeta} = n. \) Hence \( \{\eta_\alpha : \alpha \in w'\} = \{\eta_\alpha + \rho : \alpha \in w''\} \) and since \( |w'| \geq 5, \) the linear independence of \( \bar{\eta} \) implies \( \rho = 0 \) and \( w' = w'' \) and the desired conclusion follows.

RE (⑩) : Let us prove clause (⑩)10 now.

Suppose that \( m(\ell_0, w'), m(\ell_1, w'') \in \mathcal{M}, \) \( \delta \in w', \) \( |\delta \cap w'| = k(w'), \) \( \text{rk}(w') = -1, \) and \( m(\ell_0, w') \supseteq^* m(\ell_1, w''). \) Assume towards contradiction that there are \( \varepsilon_0, \varepsilon_1 \in w'' \) such that

\[
(\otimes)_0 \eta_{\varepsilon_0} |\ell_1 \neq \eta_{\varepsilon_1} |\ell_1 \text{ and } \eta_{\delta} |\ell_0 \leq \eta_{\varepsilon_0} \text{ and } \eta_{\delta} |\ell_0 < \eta_{\varepsilon_1}.
\]

Without loss of generality \( |w''| = |w'| + 1 \geq 6. \)

Since we must have \( \ell_0 < n, \) for no \( \alpha, \beta \in w' \) we can have \( h_i(\alpha, \beta) = M_\zeta. \) Therefore either \( w' \subseteq w_\xi \) or \( w' \subseteq w_\zeta. \) Also,

\[
(\otimes)_1 \text{ if } (\alpha, \beta) \in (w'')^{(2)} \setminus \{(\varepsilon_0, \varepsilon_1), (\varepsilon_1, \varepsilon_0)\} \text{ then } h_i(\alpha, \beta) < M_\xi \text{ for } i < i.
\]
Note that

\( (\otimes)_2 \) if \((\alpha, \beta) \in (w_\xi)^{(2)} \cup (w_\zeta)^{(2)} \) then \( \min(\{ \ell : \eta_\alpha(\ell) \neq \eta_\beta(\ell) \}) < n_\xi \) and there are no repetitions in the sequence \( \langle g_i(\alpha, \beta) | n_\xi, g_i(\beta, \alpha) | n_\zeta : i < \iota \rangle \).

Let \( \ell^* = \min(\ell_1, n_\xi) \).

Now, if \( w' \cup w'' \subseteq w_\xi \), then considering \( m(\ell_0, w') \) and \( m(\ell^*, w'') \) (and remembering \( (\otimes)_2 \)) we see that \( \ell_0 < n_\xi \), \( m^p_\xi(\ell_0, w') \subseteq m^p_\xi(\ell^*, w'') \) and they have the property contradicting \( (\ast)_1 \) for \( p_\xi \).

If \( w' \cup w'' \subseteq w_\zeta \), then in a similar manner we get contradiction with \( (\ast)_1 \) for \( p_\zeta \).

If \( w' \subseteq w_\xi \) and \( w'' \subseteq w_\zeta \) then one easily verifies that \( \ell_0 < n_\xi \) and \( m^p_\xi(\ell_0, w') \subseteq m^p_\xi(\ell^*, \pi^{-1}[w'']) \) provide a counterexample for \( (\ast)_1 \) for \( p_\xi \).

Similarly if \( w' \subseteq w_\zeta \) and \( w'' \subseteq w_\xi \).

Consequently, the only possibility left is that \( w'' \setminus w_\xi \neq \emptyset \neq w'' \setminus w_\zeta \) and it follows from \( (\otimes)_1 \) that \( |w'' \setminus w_\xi| = |w'' \setminus w_\zeta| = 1 \). Let \( \{ \beta_\beta \} = w'' \setminus w_\zeta \) and \( \{ \gamma_\gamma \} = w'' \setminus w_\xi \); then \( \{ \varepsilon_0, \varepsilon_1 \} = \{ \beta_\beta, \gamma_\gamma \} \).

Assume \( w' \subseteq w_\xi \) (the case when \( w' \subseteq w_\zeta \) can be handled similarly). If we had \( b \neq c \), then \( \eta_{\beta_\beta} | n_\xi = \eta_{\beta_\beta} | n_\xi \neq \eta_{\gamma_\gamma} | n_\xi \). Since \( w'' \subseteq (w_\xi \cap w_\zeta) \cup \{ \beta_\beta, \gamma_\gamma \} \) we could see that \( \ell_0 < n_\xi \) and \( m^p_\xi(\ell_0, w') \subseteq m^p_\xi(\ell^*, \pi^{-1}[w'']) \) would provide a counterexample for \( (\ast)_1 \) for \( p_\xi \). Consequently, \( b = c \) and \( \ell_1 > n_\xi \). Now, remembering \( (\otimes)_0 \), \( \eta_{\beta_\beta} | \ell_0 = \eta_{\gamma_\gamma} | \ell_0 \) and \( m^p_\xi(\ell_0, w') \supseteq m^p_\xi(\ell_0, w'') \setminus \{ \beta_\beta \}, \) so by \( (\ast)_9 \) for \( p_\xi \) we conclude

\[ \rk(\nu'' \setminus \{ \beta_\beta \}) = -1 \quad \text{and} \quad |\beta_\beta \cap (\nu'' \setminus \{ \beta_\beta \})| = k(\nu'' \setminus \{ \beta_\beta \}). \]

Let \( \zeta^* = \zeta(\nu'' \setminus \{ \beta_\beta \}) \) and \( k^* = k(\nu'' \setminus \{ \beta_\beta \}) \). For \( \varepsilon \in A \setminus \{ \xi \} \) let \( \pi^\varepsilon : w_\xi \rightarrow w_\varepsilon \) be the order isomorphism and let \( \gamma(\varepsilon) \in \pi^\varepsilon[w'' \setminus \{ \beta_\beta \}] \) be such that \( |\pi^\varepsilon[w'' \setminus \{ \beta_\beta \}] \cap \gamma| = k^* \) (necessarily \( \gamma(\varepsilon) = \pi^\varepsilon(\beta_\beta) \in w_\varepsilon \setminus w_\xi \)). Then

- \( \pi^\varepsilon[w'' \setminus \{ \beta_\beta \}] = (w'' \cap (w_\xi \cap w_\varepsilon)) \cup \{ \gamma(\varepsilon) \} = w'' \setminus \{ \beta_\beta, \beta_\gamma \} \cup \{ \gamma(\varepsilon) \} \),
- \( \rk(\pi^\varepsilon[w'' \setminus \{ \beta_\beta \}]) = -1 \), and \( \zeta(\pi^\varepsilon[w'' \setminus \{ \beta_\beta \}]) \),
- \( k(\pi^\varepsilon[w'' \setminus \{ \beta_\beta \}]) = k^* = |\pi^\varepsilon[w'' \setminus \{ \beta_\beta \}] \cap \gamma|). \)
Hence \( M \models R_{\omega',\xi^*}[w'' \setminus \{\beta_b, \gamma_b \} \cup \{\gamma(\varepsilon)\}] \) for each \( \varepsilon \in A \setminus \{\xi\} \). Consequently, the set
\[
\left\{ \alpha < \lambda : M \models R_{\omega',\xi^*}[w'' \setminus \{\beta_b, \gamma_b \} \cup \{\alpha\}] \right\}
\]
is uncountable, contradicting \((\oplus)_c\).

**RE \((*)_{11}\)**: Let us argue that \((*)_{11}\) is satisfied as well and for this suppose that \(\rho_i^0, \rho_i^1 \in \bigcup_{m < M} (t_m \cap n^2)\) (for \(i < \iota\)) are such that

- (a) there are no repetitions in \(\langle \rho_i^0, \rho_i^1 : i < \iota \rangle\), and
- (b) \(\rho_i^0 + \rho_i^1 = \rho_j^0 + \rho_j^1\) for \(i < j < \iota\).

Clearly, if all \(\rho_i^0, \rho_i^1\) are form \(\rho^\sim(\{0, \ldots, 0\})\), then we may use condition \((*)_{11}\) for \(p_\xi\) and conclude that for some \(\alpha_0, \alpha_1 \in w_\xi\) we have
\[
\{\{\rho_i^0, \rho_i^1\} : i < \iota\} = \{\{g_i(\alpha_0, \alpha_1), g_i(\alpha_1, \alpha_0)\} : i < \iota\}.
\]

So assume that we are not in the situation when all \(\rho_i^0, \rho_i^1\) are form \(\rho^\sim(\{0, \ldots, 0\})\).

Note that if \(\rho \in \bigcup_{m < M} (t_m \cap n^2)\) and \(\rho(n_\xi) = 0\), then \(\rho[n_\xi, n] = 0\).

Hence, remembering definitions in \((*)_{18}\), if \(\rho_0, \rho_1, \rho_2, \rho_3 \in \bigcup_{m < M} (t_m \cap n^2)\), \(\rho_0 + \rho_1 = \rho_2 + \rho_3\) and \(\rho_0(n_\xi) = 0\) but \(\rho_1(n_\xi) = 1\), then \(\{\rho_0, \rho_1\} = \{\rho_2, \rho_3\}\). Therefore, under current assumption, we must have \(\rho_i^0(n_\xi) = \rho_i^1(n_\xi) = 1\) for all \(i < \iota\). Define
\[
B = \{(a_\alpha, \gamma_c) : a < k \& c < \ell\},
\]
\[
C = \{(\gamma_b, \gamma_c) : b < c < \ell\},
\]
\[
D = \{(\beta_b, \gamma_c) : b < \ell \& c < \ell \& b \neq c\},
\]
\[
E = \{(\beta_b, \gamma_b) : b < \ell\}.
\]
(These four sets correspond to clauses \((*)_{18}^p - (\ast)^e_{18}\) in the definition of \(g_i\).) Clearly, \(\rho_i^0(n_\xi) = \rho_i^1(n_\xi) = 1\) implies that
\[
\rho_i^0, \rho_i^1 \in \{g_j(\varepsilon_0, \varepsilon_1), g_j(\varepsilon_1, \varepsilon_0) : (\varepsilon_0, \varepsilon_1) \in B \cup C \cup D \cup E, \ j < \iota\}.
\]
Note also that for each $d < N_0 - 1$,

(Ⅹ)\textsubscript{a} the set $\{\rho \in \bigcup_{m < M} (t_m \cap n^2) : \rho\{n\xi, n\xi + N_0 \} = \nu_d\}$ is not empty but it has at most two elements, and

(Ⅹ)\textsubscript{b} $|\{\rho \in \bigcup_{m < M} (t_m \cap n^2) : \rho\{n\xi, n\xi + N_0 \} = \nu_d\}| = 2$ if and only if $d = \Theta(b, c, i, 1)$ for some $b < c < \ell$ and $i < \iota$, and

(Ⅹ)\textsubscript{c} the set $\{\rho \in \bigcup_{m < M} (t_m \cap n^2) : \rho\{n\xi, n\xi + N_0 \} = \nu_d^*\}$ has at most one element, and

(Ⅹ)\textsubscript{d} $\{\rho \in \bigcup_{m < M} (t_m \cap n^2) : \rho\{n\xi, n\xi + N_0 \} = \nu_d^*\} = \emptyset$ if and only if $d = \Theta(b, c, i, 1)$ for some $b < c < \ell$ and $i < \iota$.

Now consider $\rho_i^0\{n\xi, n\xi + N_0\}, \rho_i^1\{n\xi, n\xi + N_0\}$ for $i < \iota$.

If for some $(i, x) \neq (j, y)$ we have $\rho_i^x\{n\xi, n\xi + N_0\} = \rho_j^y\{n\xi, n\xi + N_0\}$, then (using (Ⅹ)\textsubscript{a}–(Ⅹ)\textsubscript{d} and the linear independence of $\nu_d$'s) we must have that

$$\rho_i^0\{n\xi, n\xi + N_0\} = \rho_i^1\{n\xi, n\xi + N_0\} \quad \text{for all } i < \iota.$$ 

Thus, for every $i < \iota$ there are $b < c < \ell$ and $j < \iota$ such that

$$\{\rho_i^0, \rho_i^1\} = \{g_j(\gamma_b, \gamma_c), g_j(\gamma_c, \gamma_b)\}.$$ 

Since for $b < c < \ell$ we have

$$(g_j(\gamma_b, \gamma_c) + g_j(\gamma_c, \gamma_b))\{N_0, N_0 + \ell\} = \langle 0, \ldots, 0 \rangle^\text{−}_b \langle 1, \ldots, 1 \rangle^\text{−}_c \langle 0, \ldots, 0 \rangle^\text{−}_\ell,$$

we immediately get that (in the current situation) for some $b < c < \ell$ we have

$$\{\{\rho_i^0, \rho_i^1\} : i < \iota\} = \{\{g_i(\gamma_b, \gamma_c), g_i(\gamma_c, \gamma_b)\} : i < \iota\}.$$ 

So let us assume that $\rho_i^x\{n\xi, n\xi + N_0\} \neq \rho_j^y\{n\xi, n\xi + N_0\}$ for all distinct $(i, x), (j, y) \in \iota \times 2$. Since $\{1, \nu_0, \ldots, \nu_{N_0 - 2}\}$ are linearly independent we may use Lemma 4.3(2) to conclude that

$$\{\rho_i^0\{n\xi, n\xi + N_0\}, \rho_i^1\{n\xi, n\xi + N_0\} : i < \iota\} \subseteq \{\nu_d, \nu_d^* : d < N_0 - 1\}.$$
Consequently, we easily deduce that
\[ \{ \{ \rho_i^0, \rho_i^1 \} : i < \iota \} \subseteq \{ \{ g_i(\varepsilon_0, \varepsilon_1), g_i(\varepsilon_1, \varepsilon_0) \} : i < \iota & (\varepsilon_0, \varepsilon_1) \in B \cup D \cup E \}. \]

Using the linear independence of \( \eta_\xi^g \)'s and the definitions of \( g_i \)'s in (*) one checks that the three sets
\begin{align*}
\{ g_i(\varepsilon_0, \varepsilon_1) + g_i(\varepsilon_1, \varepsilon_0) : (\varepsilon_0, \varepsilon_1) \in B, \ i < \iota \}, \\
\{ g_i(\varepsilon_0, \varepsilon_1) + g_i(\varepsilon_1, \varepsilon_0) : (\varepsilon_0, \varepsilon_1) \in D, \ i < \iota \}, \\
\{ g_i(\varepsilon_0, \varepsilon_1) + g_i(\varepsilon_1, \varepsilon_0) : (\varepsilon_0, \varepsilon_1) \in E, \ i < \iota \}
\end{align*}
are pairwise disjoint. Therefore, \( \{ \{ \rho_i^0, \rho_i^1 \} : i < \iota \} \) must be included in (exactly) one of the sets
\begin{align*}
\{ \{ g_i(\varepsilon_0, \varepsilon_1), g_i(\varepsilon_1, \varepsilon_0) \} : i < \iota & (\varepsilon_0, \varepsilon_1) \in B \}, \\
\{ \{ g_i(\varepsilon_0, \varepsilon_1), g_i(\varepsilon_1, \varepsilon_0) \} : i < \iota & (\varepsilon_0, \varepsilon_1) \in D \}, \text{ or} \\
\{ \{ g_i(\varepsilon_0, \varepsilon_1), g_i(\varepsilon_1, \varepsilon_0) \} : i < \iota & (\varepsilon_0, \varepsilon_1) \in E \}.
\end{align*}
But now we easily check that for some \( (\varepsilon_0, \varepsilon_1) \in B \cup D \cup E \) we must have
\[ \{ \{ \rho_i^0, \rho_i^1 \} : i < \iota \} = \{ \{ g_i(\varepsilon_0, \varepsilon_1), g_i(\varepsilon_1, \varepsilon_0) \} : i < \iota \}. \]
This completes the verification that \( q = (w, n, M, \bar{\eta}, \bar{t}, \bar{r}, \bar{h}, \bar{g}, \mathcal{M}) \in \mathbb{P} \), and clearly \( q \) is stronger than both \( p_\xi \) and \( p_\eta \). \( \square \)

Define \( \mathbb{P} \)-names \( T_m \) and \( \eta_\alpha \) (for \( m < \omega \) and \( \alpha < \lambda \)) by
\begin{align*}
\forces_{\mathbb{P}} "T_m = \bigcup \{ t^p_m : p \in G_{\mathbb{P}} \land m < M_p \} ", \text{ and} \\
\forces_{\mathbb{P}} "\eta_\alpha = \bigcup \{ \eta^p_\alpha : p \in G_{\mathbb{P}} \land \alpha \in w^p \} ".
\end{align*}

**Claim 4.4.4.** 1. For each \( m < \omega \) and \( \alpha < \lambda \),
\[ \forces_{\mathbb{P}} "\eta_\alpha \in \omega^2 \text{ and } T_m \subseteq \omega^2 \text{ is a tree without terminal nodes}". \]

2. \( \forces_{\mathbb{P}} "\bigcup_{m < \omega} \text{lim}(T_m) \text{ is a 2}-\text{npots set}". \]

**Proof of the Claim.** (1) By Claim 4.4.2 (and the definition of the order in \( \mathbb{P} \)).

(2) Let \( G \subseteq \mathbb{P} \) be a generic filter over \( \mathbb{V} \) and let us work in \( \mathbb{V}[G] \).

Let \( k = 2\iota \) and \( \bar{T} = ((T_m)^G : m < \omega) \).

Suppose towards contradiction that \( B = \bigcup_{m < \omega} \text{lim} ( (T_m)^G ) \) is a \( k \)-\text{pots} set. Then, by Proposition 3.11, \( \text{NDRK}(\bar{T}) = \infty \). Using Lemma 3.10(5), by induction on \( j < \omega \) we choose \( m_j, m^*_j \in M_{\bar{T},k} \) and \( p_j \in G \) such that
\begin{enumerate}
\item \( \text{ndrk}(m_j) \geq \omega_1, \ |um_j| > 5 \) and \( m_j \subseteq m^*_j \subseteq m_{j+1} \),
\end{enumerate}
(ii) for each $\nu \in u_{m_j^*}$ the set $\{\eta \in u_{m_{j+1}} : \nu < \eta\}$ has at least two elements,

(iii) $p_j \leq p_{j+1}$, $\ell_{m_j} \leq \ell_{m_j^*} = n^{P_j} < \ell_{m_{j+1}}$ and $\text{rng}(n_{i|m_j}) \subseteq M^{P_j}$ for all $i < \iota$, and

(iv) $|\{\eta | n^{P_j} : \eta \in u_{m_{j+1}}\}| = |u_{m_j}| = |u_{m_j^*}|$.

Then, by (iii)+(iv), $m_j, m_j^* \in M^{P_j}_{\nu_{j},k}$. It follows from Claim 4.4.1 that for some $w_j \subseteq w_{P_j}$ and $p_j \in n^{P_j} 2$ we have $(m_j^* + p_j) \models m_{P_j}(n^{P_j}, w_j) \in M^{P_j}$.

Fix $j$ for a moment and consider $m_{P_j}(n^{P_j}, w_j) \in M^{P_j}$ and $m_{P_j+1}(n^{P_j+1}, w_{j+1}) \in M^{P_j+1}$. Since

$$(m_j^* + (p_{j+1} n^{P_j})) \models (m_{j+1}^* + p_{j+1}) \models m_{P_j+1}(n^{P_j+1}, w_{j+1}),$$

we may choose $w_j^* \subseteq w_{j+1}$ such that

$$(m_j^* + (p_{j+1} n^{P_j})) \models m_{P_j+1}(n^{P_j}, w_j^*) \models m_{P_j+1}(n^{P_j+1}, w_{j+1})$$

(and the latter two belong to $M^{P_j+1}$). Then also

$$m_{P_j+1}(n^{P_j}, w_j^*) \models m_{P_j}(n^{P_j}, w_j) + (p_j + p_{j+1} n^{P_j})$$

so by clause $(*)_9$ for $p_{j+1}$ we have

$$\text{rk}(w_j^*) = \text{rk}(w_j).$$

Clause (ii) of the choice of $m_{j+1}$ implies that

$$(\forall \gamma \in w_j^* \exists \delta \in w_{j+1} \setminus w_j^*)(w_{j+1}^* n^{P_j} = \eta_{P_j+1} n^{P_j}).$$

Let $\delta(\gamma)$ be the smallest $\delta \in w_{j+1} \setminus w_j^*$ with the above property and let $w_j^*(\gamma) = (w_j^* \setminus \{\gamma\}) \cup \{\delta(\gamma)\}$. Then, for $\gamma \in w_j^*$, $m_{P_j+1}(n^{P_j}, w_j^*(\gamma)) \in M^{P_j+1}$ and

$$m_{P_j+1}(n^{P_j}, w_j^*(\gamma)) \models m_{P_j+1}(n^{P_j}, w_j^*(\gamma)) \models m_{P_j+1}(n^{P_j+1}, w_{j+1}).$$

So by clause $(*)_9$ we know that for each $\gamma \in w_j$:

$$\text{rk}(w_j^*(\gamma)) = \text{rk}(w_j^*), \ \ \ \zeta(w_j^*(\gamma)) = \zeta(w_j^*), \ \ \ \text{and} \ \ \ k(w_j^*(\gamma)) = k(w_j^*).$$
Let $n = |w_j^*|$, $\zeta = \zeta(w_j^*)$, $k = k(w_j^*)$, and let $w_j^* = \{\alpha_0, \ldots, \alpha_k, \ldots, \alpha_{n-1}\}$ be the increasing enumeration. Let $\alpha_k^* = \delta(\alpha_k)$. Then clause $(\star)_9$ also gives that $w_j^*(\alpha_k) = \{\alpha_0, \ldots, \alpha_{k-1}, \alpha_k^*, \alpha_{k+1}, \ldots, \alpha_{n-1}\}$ is the increasing enumeration. Now,

\[
\mathbb{M} \models R_{n,\zeta}[\alpha_0, \ldots, \alpha_{k-1}, \alpha_k, \alpha_{k+1}, \ldots, \alpha_{n-1}] \quad \text{and} \\
\mathbb{M} \models R_{n,\zeta}[\alpha_0, \ldots, \alpha_{k-1}, \alpha_k^*, \alpha_{k+1}, \ldots, \alpha_{n-1}],
\]

and consequently if $\text{rk}(w_j^*) \geq 0$, then

\[
\text{rk}(w_{j+1}) \leq \text{rk}(w_j^* \cup \{\alpha_k^*\}) < \text{rk}(w_j^*) = \text{rk}(w_j)
\]

(remember $(\otimes)_4$).

Now, unfixing $j$, suppose that we constructed $w_{j+1}, w_j^*$ for all $j < \omega$. It follows from our considerations above that for some $j_0 < \omega$ we must have:

(a) $\text{rk}(w_{j_0}^*) = -1$, and

(b) $m^{P_{j_0}+1}(n^{P_{j_0}}, w_{j_0}^*) \sqsupset^* m^{P_{j_0}+1}(n^{P_{j_0}+1}, w_{j_0+1})$

(and both belong to $\mathcal{M}^{P_{j_0}+1}$),

(c) for every $\alpha \in w_{j_0}^*$ we have

\[
\left| \{ \beta \in w_{j_0+1} : \eta_{\alpha}^{P_{j_0}+1} \sqsubset \eta_{\beta}^{P_{j_0}+1} \} \right| > 1.
\]

However, this contradicts clause $(\star)_{10}$ (for $p_{j_0+1}$). □

**Corollary 4.5.** Assume MA and $\aleph_\alpha < c$, $\alpha < \omega_1$. Let $3 \leq \iota < \omega$. Then there exists a $\Sigma^0_2$ 2\iota-\textit{npots} -set $B \subseteq \omega^2$ which has $\aleph_\alpha$ many pairwise $2\iota$–nondisjoint translations.

**Proof.** Standard modification of the proof of Theorem 4.4. □

**Corollary 4.6.** Assume $\text{NPr}_{\omega_1}(\lambda)$ and $\lambda = \lambda^{\aleph_0} < \mu = \mu^{\aleph_0}$, $3 \leq \iota < \omega$. Then there is a ccc forcing notion $Q$ of size $\mu$ forcing that

(a) $2^{\aleph_0} = \mu$ and

(b) there is a $\Sigma^0_2$ 2\iota-\textit{npots} -set $B \subseteq \omega^2$ which has $\lambda$ many pairwise $2\iota$–nondisjoint translates but not $\lambda^+$ such translates.
Proof. Let $\mathbb{P}$ be the forcing notion given by Theorem 4.4 and let $\mathbb{Q} = \mathbb{P} \ast C_{\mu}$. Use Proposition 3.3(4) to argue that the set $B$ added by $\mathbb{P}$ is a npots-set in $V^{\mathbb{Q}}$. By 3.3(3) this set cannot have $\lambda^+$ pairwise $2\tau$-nondisjoint translates, but it does have $\lambda$ many pairwise $2\tau$-nondisjoint translates (by absoluteness).

Remark 4.7. It follows from Proposition 3.3(1,2), that if there exists a $\Sigma^0_2$ pots-set $B \subseteq \omega^2$ such that for some set $A \subseteq \omega^2$ we have $(B + a) \cap (B + b) \neq \emptyset$ for all $a, b \in A$, then std$(B) \subseteq \omega^2 \times \omega^2$ is a $\Sigma^0_2$ set which contains a $|A|$-square but no perfect square. Thus Corollary 4.6 is a slight generalization of Shelah [7, Theorem 1.13].

5. Further research

The case of $k = 4$ in Theorem 4.4 will be dealt with in a subsequent paper [6] alongside with further investigations of $\Sigma^0_2$ subsets of $\omega^2$ with pregiven rank NDRK. In subsequent works we will also investigate the general case of Polish groups (not just $\omega^2$). The following two problems are still open however.

Problem 5.1. Is is consistent to have a Borel set $B \subseteq \omega^2$ such that

- for some uncountable set $H$, $(B + x) \cap (B + y)$ is uncountable for every $x, y \in H$, but

- for every perfect set $P$ there are $x, y \in P$ with $(B + x) \cap (B + y)$ countable?

Problem 5.2. Is it consistent to have a Borel set $B \subseteq \omega^2$ such that

- $B$ has uncountably many pairwise disjoint translations, but

- there is no perfect of pairwise disjoint translations of $B$?

References

Proof. Let $P$ be the forcing notion given by Theorem 4.4 and let $Q = P^* C_{\mu}$. Use Proposition 3.3(4) to argue that the set $B$ added by $P$ is a $\text{npots}$–set in $V_Q$. By 3.3(3) this set cannot have $\lambda$ pairwise 2-$\iota$–nondisjoint translates, but it does have $\lambda$ many pairwise 2-$\iota$–nondisjoint translates (by absoluteness). □

Remark 4.7. It follows from Proposition 3.3(1,2), that if there exists a $\Sigma_0^2$–set $B \subseteq \omega^2$ such that for some set $A \subseteq \omega^2$ we have $(B + a) \cap (B + b) \neq \emptyset$ for all $a, b \in A$, then $\text{stnd}(B) \subseteq \omega^2 \times \omega^2$ is a $\Sigma_0^2$-set which contains a $|A|$–square but no perfect square. Thus Corollary 4.6 is a slight generalization of Shelah [7, Theorem 1.13].

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