ERRATA ON “ON THE VARIETY OF HEYTING ALGEBRAS WITH SUCCESSOR GENERATED BY ALL FINITE CHAINS”

In [3] we have claimed that finite Heyting algebras with successor only generate a proper subvariety of that of all Heyting algebras with successor, and in particular all finite chains generate a proper subvariety of the latter. As Xavier Caicedo made us notice, this claim is not true. He proved, using techniques of Kripke models, that the intuitionistic calculus with \( S \) has finite model property and from this result he concluded that the variety of Heyting algebras with successor is generated by its finite members [2].

This fact particularly affects Section 3.2 of our article. Concretely, in Remark 3.3, our claim “Let \( \mathcal{K} \) be a class of \( S \)-Heyting algebras of height less or equal to a fixed ordinal \( \xi \). Using the categorical duality between \( S \)-Heyting algebras and \( S \)-Heyting spaces, it can be shown that the elements of classes \( H(\mathcal{K}) \), \( S(\mathcal{K}) \) and \( P(\mathcal{K}) \) have also height less or equal to \( \xi \). Here \( H \), \( S \) and \( P \) are the class operators of universal algebra. Hence for each ordinal \( \xi \), the class of \( S \)-Heyting algebras of height less or equal to \( \xi \) is a variety” is not true as stated. It remains valid only if \( \xi \) is a finite ordinal.

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In particular, the class of $S$-Heyting algebras of height $\omega$ is not a variety and the variety generated by all finite chains is exactly the variety of linear $S$-Heyting algebras.

In what follows, instead of using the proof given in [2], which is not published, we shall give a simple algebraic proof that the variety of linear Heyting algebras with $S$ is generated by the finite chains.

Let $T$ be the type of Heyting algebras with successor built in the usual way from the operation symbols $\land, \lor, \rightarrow$, and $S$ corresponding to meet, join, implication and successor, respectively. Write $T(X)$ for the term algebra of type $T$ with variables in the set $X$. It is well known that any function $v : X \rightarrow H$, with $H$ a $S$-Heyting algebra, may be extend to a unique homomorphism $v : T(X) \rightarrow H$.

Write SLH for the variety of linear $S$-Heyting algebras. Recall that SLH is said to have the finite model property (FMP) if for every $\varphi \in T(X)$ there is a linear $S$-Heyting algebra $H$ and a homomorphism $v : T(X) \rightarrow H$ such that if $v(\varphi) \neq 1$ then there is a finite linear $S$-Heyting algebra $L$ and a homomorphism $w : T(X) \rightarrow L$ such that $w(\varphi) \neq 1$. Let us prove that SLH has the FMP. In so doing we shall use the two following well known Lemmata.

**Lemma 1.** (Lemma 1.1 of [4]) If $P$ is a prime filter in a linear algebra $H$, then $H/P$ is a chain.

**Lemma 2.** Let $C$ be a $S$-Heyting algebra which is a chain and $L$ a bounded sublattice of $C$, endowed with its implication $\rightarrow_L$ and successor $S_L$, as finite lattice. Then, we have that,

1. If $x, y \in L$ then $x \rightarrow y = x \rightarrow_L y$.
2. If $x, S(x) \in L$ then $S_L(x) = S(x)$.

Take $\alpha$ and $\beta$ in $T(X)$. Note that an equation $\alpha \approx \beta$ holds in a $S$-Heyting algebra $H$ if and only if $\alpha \rightarrow \beta \approx 1$ holds in $H$; and the latter is equivalent to ask that for any homomorphism $v : T(X) \rightarrow H$, $v(\alpha \rightarrow \beta) = 1$.

We are now ready to prove the main result.

**Proposition 3.** The variety SLH has the FMP.
Proof. Let $\psi \in T(X)$, $H$ be a linear $S$-Heyting algebra and $v : T(X) \to H$ be a homomorphism such that $v(\psi) \neq 1$. Let $\to$ and $S$ be the implication and the successor of $H$ respectively. We will find a finite chain $L$ and a homomorphism $t : T(X) \to L$ such that $t(\psi) \neq 1$.

By the Prime Filter Theorem there is a prime filter $P$ of $H$ such that $v(\psi) \notin P$, so $v(\psi)/P \neq 1$. Write $C$ in place of $H/P$. Hence, by Lemma 1, we have that $C$ is a chain. On the other hand, using that the successor operator is compatible [1] we have that the quotient function $\rho : H \to C$ is a homomorphism. Hence $w = \rho v : T(X) \to C$ is a homomorphism. Note that $w(\psi) = v(\psi)/P \neq 1$.

Let $\text{Sub}_\psi = \{\psi_1, ..., \psi_n\}$ be the set of subformulas of $\psi$ and $L$ the subset of $C$ given by $\{0, 1\} \cup \{w(\alpha) : \alpha \in \text{Sub}_\psi\}$. If $V$ is the set of propositional variables which appear in $\psi$, we can define the function $t : X \to L$ in the following way:

$$t(x_i) = \begin{cases} w(x_i) & \text{if } x_i \in V \\ 0 & \text{if } x_i \notin V. \end{cases}$$

We know that $t$ may be uniquely extended to a homomorphism $t : T(X) \to L$. Using Lemma 2, we can prove, by an easy induction on sentences, that $t(\psi_i) = w(\psi_i)$ for $i = 1, ..., n$. Therefore we have that $t(\psi) = w(\psi) \neq 1$. \hfill $\square$

In particular, we have that the following corollary holds.

**Corollary 4.** The variety SLH is generated by all finite chains.

Finally, we want to call the attention to a typos in Proposition 5.6. We wrote SHS in place of SLH.

References


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