FINITELY BASED MONOIDS OBTAINED FROM
NON-FINITELY BASED SEMIGROUPS

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Abstract. Presently, no example of non-finitely based finite semigroup $S$ is known for which the monoid $S^1$ is finitely based. Based on a general result of M. V. Volkov, two methods are established from which examples of such semigroups can be constructed.

1. Introduction. A semigroup is finitely based if the identities it satisfies are finitely axiomatizable. Commutative semigroups $^9$, idempotent semigroups $^3,^5$, and finite groups $^8$ are finitely based, but not all semigroups are finitely based in general. Further, the class $\mathfrak{FB}$ of finitely based semigroups is not closed under common operators such as the formation of homomorphic images, subsemigroups, and direct products. Refer to the survey by Volkov $^{14}$ for more information on these operators and the finite basis problem for semigroups in general. The present article is concerned with the operator that maps each semigroup $S$ to the smallest containing monoid

$$S^1 = \begin{cases} S & \text{if } S \text{ is a monoid,} \\ S \cup \{1\} & \text{otherwise.} \end{cases}$$

The class $\mathfrak{FB}$ is not closed under this operator; there exist finitely based semigroups $S$ such that the monoids $S^1$ are non-finitely based. The earliest example demonstrating this property, published by Perkins $^9$ in 1969, is a certain semigroup $R_{24}$ of order 24; see Section $^3$. Perkins’s work in fact contains a much smaller example that he was unaware of at that time: he proved that the Brandt monoid $B^1_2$ is non-finitely based $^9$, while the Brandt semigroup

$$B_2 = \langle a, b \mid a^2 = b^2 = 0, aba = a, bab = b \rangle$$

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of order five was later shown by Trahtman to be finitely based \[12\]. These examples led Shneerson \[11\] to question the existence of semigroups having the “opposite” property.

**Question 1.** Do non-finitely based semigroups $S$ exist for which the monoids $S^1$ are finitely based?

In what follows, it is convenient to call a semigroup $S$ conformable if $S$ is non-finitely based while $S^1$ is finitely based. Shneerson provided an affirmative answer to Question 1 by proving that the semigroup

$$T = \langle a, b \mid aba = ba \rangle$$

is conformable \[11\]. However, unlike the finite examples $B_2$ and $R_{24}$ that motivated Shneerson’s question, the semigroup $T$ is infinite. Apart from $T$, no other semigroup has since been found to be conformable. Therefore, the restriction of Question 1 to finite semigroups is of fundamental interest.

**Question 2.** Do finite conformable semigroups exist?

Recall that a semigroup $S$ with zero 0 is nilpotent if there exists some $n \geq 1$ such that the product of any $n$ elements of $S$ equals 0. Each nilpotent semigroup satisfies the identity

$$x_1x_2\cdots x_n = y_1y_2\cdots y_n$$

for some $n \geq 1$ and so is easily shown to be finitely based \[9\]. It turns out that by the following general result of Volkov \[13\], which was established prior to Question 1 being posed by Shneerson, an abundance of finite conformable semigroups can be constructed from nilpotent semigroups.

**Lemma 3.** Suppose that $N$ is any nilpotent semigroup. Then for any semigroup $S$, the direct product $S \times N$ is finitely based if and only if $S$ is finitely based.

2. **Constructing finite conformable semigroups.** Recall that the variety generated by a semigroup $S$, denoted by $\text{var} S$, is the smallest class of semigroups containing $S$ that is closed under the formation of homomorphic images, subsemigroups, and arbitrary direct products. A semigroup $S$ satisfies the same identities as the variety $\text{var} S$ it generates \[2\].

**Theorem 4.** Suppose that $S$ and $N$ are any semigroups such that

(a) $S^1$ is non-finitely based;
(b) $N$ is nilpotent;
(c) $S^1 \times N^1$ is finitely based.

Then the direct product $P = S^1 \times N$ is conformable.
Proof. The semigroup $P$ is non-finitely based by (a), (b), and Lemma 3. Since $P$ is a subsemigroup of $S^1 \times N^1$, it belongs to the variety $\text{var}(S^1 \times N^1)$. The inclusion $\text{var} P^1 \subseteq \text{var}(S^1 \times N^1)$ then follows [1, Lemma 7.1.1]. But the monoids $S^1$ and $N^1$ are embeddable in $P^1$ so that $\text{var} P^1 = \text{var}(S^1 \times N^1)$. Therefore, the monoid $P^1$ is finitely based by (c).

Theorem 5. Suppose that $S$ and $N$ are any semigroups such that

(a) $S^1$ is non-finitely based;
(b) $N$ is nilpotent;
(c) $N^1$ is finitely based;
(d) $\text{var} S^1 \subseteq \text{var} N^1$.

Then the direct product $P = S^1 \times N$ is conformable.

Proof. Following the proof of Theorem 4, the semigroup $P$ is non-finitely based with $\text{var} P^1 = \text{var}(S^1 \times N^1)$. Then (d) implies that $\text{var} P^1 = \text{var} N^1$, whence the monoid $P^1$ is finitely based by (c).

The following results of Jackson and Sapir [6] now provide the appropriate finite semigroups $S$ and $N$ to construct the conformable semigroups $P$ in Theorems 4 and 5.

Lemma 6. There exist finite nilpotent semigroups $S$ and $N$ such that $S^1$ and $N^1$ are non-finitely based while $S^1 \times N^1$ is finitely based.

Lemma 7. There exist finite nilpotent semigroups $S$ and $N$ such that $S^1$ is non-finitely based, $N^1$ is finitely based, and $\text{var} S^1 \subseteq \text{var} N^1$.

Jackson and Sapir in fact presented methods for locating as many of the semigroups in Lemmas 6 and 7 as desired [6, Corollaries 3.1 and 5.2].

3. Explicit examples of finite conformable semigroups. Let $A^+$ denote the free semigroup over a countably infinite alphabet $A$. Elements of $A^+$ are called words. For any finite set $W = \{w_1, \ldots, w_k\}$ of words, let $R(w_1, \ldots, w_k)$ denote the Rees quotient of $A^+$ over the ideal of all words that are not factors of any word in $W$. Equivalently, $R(w_1, \ldots, w_k)$ can be treated as the semigroup that consists of every nonempty factor of every word in $W$, together with a zero element 0, with binary operation $\cdot$ given by

$$u \cdot v = \begin{cases} uv & \text{if } uv \text{ is a factor of some word in } W, \\ 0 & \text{otherwise.} \end{cases}$$

It is easily seen that the semigroup $R(w_1, \ldots, w_k)$ is nilpotent. The semigroup $R_{24}$ of Perkins introduced in Section 1 is $R(xyzyx, xzyxy, xyxy, xzx)$.

Consider the semigroups

$$R_8 = R(xyxy), \quad R_{12} = R(xxyy, xyyx), \quad \text{and} \quad R_{15} = R(xyxy, xxyy, xyyx)$$

where $|R_8| = 8$, $|R_{12}| = 12$, and $|R_{15}| = 15$. Then
• $R_8^1$ is non-finitely based \([6, \text{Example 4.2}]\);
• $R_8^{12}$ is non-finitely based \([6, \text{proof of Corollary 5.1}]\);
• $R_8^{15}$ is finitely based \([6, \text{Corollary 3.2 and proof of Corollary 5.1}]\);
• $\var(R_8^1 \times R_8^{12}) = \var R_8^{15}$ \([6, \text{Lemma 5.1}]\).

It follows that the pairs $(S, N) = (R_8, R_{12})$ and $(S, N) = (R_8, R_{15})$ satisfy Lemmas [6] and [7], respectively. Therefore, by Theorems [4] and [5], the semigroups $R_8^1 \times R_{12}$ and $R_8^1 \times R_{15}$ are conformable.

Now since the conformable semigroup $P = S^1 \times N$ in Theorems \([4, 5]\) is a direct product, its order $|S^1||N|$ can be quite large in general. But it turns out that the semigroup $P$ contains a proper subsemigroup that is also conformable. Define

$$P_* = S_*^1 \cup N_*$$

where $S_*^1 = \{(a, 0) \mid a \in S^1\}$ and $N_* = \{(0, b) \mid b \in N\}$. Then it is easily seen that $S_*^1$, $N_*$, and $P_*$ are subsemigroups of $P$.

**Proposition 8.** The semigroup $P_*$ is conformable.

**Proof.** The isomorphic relations $S^1 \cong S_*^1$ and $N \cong N_*$ clearly hold. Therefore,

$$\var P = \var (S^1 \times N) = \var (S_*^1 \times N_*) \subseteq \var P_* \subseteq \var P,$$

whence the semigroups $P$ and $P_*$ generate the same variety and so satisfy the same identities. The result thus follows. \[\square\]

The semigroup $P_*$ has order $|S_*^1| + |N_*| - 1$ and so is often much smaller than the semigroup $P$ with order $|S^1||N|$. For instance,

$$\left\{ (|P_*|, |P|) \mid (S, N) = (R_8, R_{12}) \right\} = \left\{ (20, 108) \right\} \text{ if (S, N) = (R_8, R_{12}),}$$

$$\left\{ (23, 135) \right\} \text{ if (S, N) = (R_8, R_{15}).}$$

On the other hand, the semigroup $P_*$ is still quite large; the order of any non-finitely based monoid of the form $R(w_1, \ldots, w_k)$ is at least nine \([6, \text{Theorem 4.3}]\) so that $|P_*| \geq 9 + 2 - 1 = 10$.

In view of the small semigroup $B_2$ that motivated Question [1] it is natural to pose the following question:

**Question 9.** What is the smallest possible order of a conformable semigroup?

Based on results of Lee *et al.* \([7]\), Sapir \([10]\), and Zhang \([15]\), the order of any conformable semigroup is at least seven.

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