ASYMPTOTIC ESTIMATE OF ABSOLUTE PROJECTION CONSTANTS

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Abstract. In this note we construct a sequence of real, $k$-dimensional symmetric spaces $Y_k$ satisfying
\[ \liminf_k \lambda_k^S \sqrt{k} \geq \liminf_k \lambda(Y_k, l_1)/\sqrt{k} > \max_{w \in [0, a_2]} h(w) > 1/(2 - \sqrt{2}/\pi), \]
where $\lambda_k^S$ is defined in (4) and
\[ h(w) = a_2^2 \sqrt{2/\pi} + 2a_1 \sqrt{a_2^2 - w^2} + w \sqrt{a_2^2 - w^2} \]
with $a_1 = 1/(2 - \sqrt{2}/\pi)$ and $a_2 = 1 - a_1$. This improves the lower bound obtained in [3], Th. 5.3 by $\max_{w \in [0, a_2]} h(w)$.

1. Introduction. Let $X$ be a normed space and let $V$ be a linear subspace of $X$. Denote by $\mathcal{P}(X, V)$ the set of all projections from $X$ onto $V$, i.e., the set of all continuous extensions of $id : V \to V$ to $X$. Let
\[ \lambda(V, X) = \inf \{ \|P\| : P \in \mathcal{P}(X, V) \} \]
and
\[ \lambda(V) = \sup \{ \lambda(V, X) : V \subset X, \text{ as Banach spaces} \}. \]
We call $\lambda(V, X)$ the relative projection constant of $V$ in $X$ and $\lambda(V)$ the absolute projection constant of $V$. A projection $P \in \mathcal{P}(X, V)$ is called minimal if $\|P\| = \lambda(V, X)$. Let us denote
\[ \lambda_k = \sup \{ \lambda(Y) : Y \text{ is a real, } k\text{-dimensional space} \}. \]

2010 Mathematics Subject Classification. 47A58, 41A65.

Key words and phrases. Absolute projection constant, minimal projection, symmetric spaces.
It is known (see e.g. [8]), by the compactness of the Banach–Mazur compactum and the continuity of the function $X \to \lambda(X)$, that there exists a $k$-dimensional, real space $X^k$ such that

\[(1) \quad \lambda_k = \lambda(X^k).\]

Moreover, $X^k$, as a separable Banach space, is isometric to a subspace of $l^\infty$ and $\lambda(X^k) = \lambda(X^k, l^\infty)$ (see e.g. [11]). By the Kadeč–Snobar Theorem [7], $\lambda_k \leq \sqrt{k}$. Moreover, the examples from [4] show that this estimate is asymptotically the best possible, which means that

\[(2) \quad \lim_{k} \lambda(X^k)/\sqrt{k} = 1,\]

where $X^k$ is given by [1]. For other related results see [2,4–6].

It is worth saying that the spaces $X^k$ defined by (1) are not symmetric. Recall that a $k$-dimensional real Banach space $V$ is called symmetric if there is a basis $v_1, \ldots, v_k$ of $V$ such that

\[(3) \quad \| \sum_{j=1}^{k} a_j v_j \| = \| \sum_{j=1}^{k} \epsilon_j a_{\sigma(j)} v_j \|\]

for any $a_1, \ldots, a_k \in \mathbb{R}$, $\epsilon_k \in \{-1, 1\}$ and $\sigma \in \Sigma_k$, where $\Sigma_k$ denotes the set of all permutations of $\{1, \ldots, k\}$. Moreover, equality (2) does not hold in the case of symmetric spaces, which has been shown in [6]. It has been proven in [6] that

$$\limsup_{k} \frac{\lambda_k^S/\sqrt{k}}{\sqrt{k}} < 1 - 1/900,$$

where

\[(4) \quad \lambda_k^S = \sup \{ \lambda(Y) : Y \text{ real, } k\text{-dimensional, symmetric space} \}.\]

It also has been conjectured in [6], p. 36, that

\[(5) \quad \limsup_{k} \frac{\lambda_k^S/\sqrt{k}}{\sqrt{k}} = 1/(2 - \sqrt{2/\pi}).\]

This conjecture has been partially motivated by [5], Prop. 2, where the existence of $k$-dimensional, real, symmetric spaces $Y^k$ satisfying

\[(6) \quad \limsup_{k} \lambda(Y^k)/\sqrt{k} = 1/(2 - \sqrt{2/\pi})\]

has been shown. Observe that by [10]

\[(7) \quad \lim_{k} \lambda(l_2^k)/\sqrt{k} = \sqrt{2/\pi}.\]

Since

$$\sqrt{2/\pi} = 0.7979 \ldots < 1/(2 - \sqrt{2/\pi}) = 0.8319 \ldots,$$
the spaces $Y^k$ have asymptotically larger absolute projection constants than the Euclidean spaces $l_2^k$. Also in [8], the Marcinkiewicz spaces satisfying (6) have been constructed.

Above conjecture (5) has been disproved in [3], Th. 5.3. Moreover, in [1], for $k \geq 3$, there have been constructed symmetric $k$-dimensional subspaces $V^k$ of $l_1$ having a very simple structure such that

$$\lim_k \lambda(V^k, l_1)/\sqrt{k} = 1/(2 - \sqrt{2/\pi}).$$

The aim of this note is to show the existence of $k$-dimensional, real and symmetric subspaces $V^k$ satisfying

$$\lim_k \frac{\lambda_k^2}{\sqrt{k}} \geq \liminf_k \lambda(Y^k, l_1)/\sqrt{k} > \max_{w \in [0, a_2]} h(w) > 1/(2 - \sqrt{2/\pi}),$$

where $h(w) = a_1^2 \sqrt{2/\pi} + 2a_1 \sqrt{a_2^2 - w^2} + w \sqrt{a_2^2 - w^2}$ with $a_1 = \frac{1}{2 - \sqrt{2/\pi}}$ and $a_2 = 1 - a_1$. This improves the lower bound obtained in [3], Th. 5.3 by $\max_{w \in [0, a_2]} h(w)$.

2. Auxiliary results. In this section we present some definitions and results which will be of use later.

**Definition 2.1.** Let $x \in \mathbb{R}^k$ and let $\Sigma_k$ denote the set of all permutations of $\{1, \ldots, k\}$. Suppose $J : \Sigma_k \times \{-1,1\}^k \rightarrow \{1, \ldots, 2^k k!\}$ is a fixed bijection such that $J(id, (1, \ldots, 1)) = 1$. By $[[x]]$ we denote the $k \times 2^k k!$ matrix with the columns $x^{(j)}$, $j = 1, \ldots, 2^k k!$, where

$$x^{(j)} = (\epsilon \circ \sigma)(x) = (\epsilon_1 x_{\sigma(1)}, \ldots, \epsilon_k x_{\sigma(k)}).$$

Here $\epsilon \in \{-1,1\}^k$, $\sigma \in \Sigma_k$ are so chosen that $j = J(\sigma, \epsilon)$. Observe that, for any $x \in \mathbb{R}^k$, $x^{(1)} = x$. We will refer to the matrix $[[x]]$ as the block generated by $x$.

**Definition 2.2.** Let $n, N \in \mathbb{N}$. Put $n = N 2^k k!$. Let $x^1, \ldots, x^N \in \mathbb{R}^k$. A linear subspace $V \subseteq l_1^n$ is said to be generated by $(x^1, \ldots, x^N)$ if and only if the rows $v^1, \ldots, v^k$ of the $k \times N 2^k k!$ matrix $[[x^1]], \ldots, [[x^N]]$ form a basis of $V$, where, for $i = 1, \ldots, N$, $[[x^i]]$ is the block generated by $x^i$ (see Def. 2.1). It is easy to check that $V$ is a symmetric space (see (3)) with respect to $v^1, \ldots, v^k$.

The following notation will also be used. For $x, y \in \mathbb{R}^k$, set

$$y \cdot [[x]] = \sum_{i=1}^{2^k k!} \left( \sum_{j=1}^k y_j x_j^{(i)} \right) = \sum_{(\sigma, \epsilon) \in \Sigma_k \times \{-1,1\}^k} \left| \sum_{j=1}^k y_j \epsilon_j x_{\sigma(j)} \right|.$$

Observe that for any $x, y \in \mathbb{R}^k$

$$y \cdot [[x]] = x \cdot [[y]].$$

(8)
Now let $V \subset l_1^{(n)}$ be a subspace generated by $x^1, \ldots, x^N$ from $\mathbb{R}^k$. For $z \in \mathbb{R}^k$ we set
\[
\|z\| = \|\sum_{j=1}^{k} z_j v^j\|_1,
\]
where $v^1, \ldots, v^k$ is the basis of $V$ associated with $x^1, \ldots, x^N$ by Def. (2.2). By $\|z\|_e$ we denote the Euclidean norm of $z$. Observe that
\[
\|z\| = \sum_{j=1}^{N} z \cdot [[x^j]].
\]
The main tool for our investigations will be the following theorem proved in [3], Th. 4.1. We present here a version of it more convenient for our purposes.

**Theorem 2.1.** Let $k, N \in \mathbb{N}$. Let $n = N2^kk!$. Consider the following extremal problem. Maximize the function $f : \mathbb{R}^{kN} \to \mathbb{R}$ defined for $x^1, \ldots, x^N \in \mathbb{R}^k$ by
\[
(9) \quad f(x^1, \ldots, x^N) = \sum_{j=1}^{N} \|x^j\|
\]
under the conditions
\[
\sum_{j=1}^{N} \|x^j\|_e = 1.
\]
If $f$ attains its maximum at $(y^1, \ldots, y^N)$ then the symmetric $k$-dimensional space $Y^N$ generated by $y^1, \ldots, y^N$ satisfies
\[
\frac{f(y^1, \ldots, y^N)}{2^{k(k-1)!}} = \lambda(Y^N, l_1) \leq \lambda(Y^N).
\]

3. Main result. In this section we show that there exist $k$-dimensional maximal symmetric spaces $V^k$ satisfying
\[
(10) \quad \liminf_k \frac{\lambda(Y^k, l_1)}{\sqrt{k}} > \max_{w \in [0, a_2]} h(w) > 1/(2 - \sqrt{2/\pi})
\]
where $h(w) = a_1^2 \sqrt{2/\pi} + 2a_1 \sqrt{a_2^2 - w^2} + w \sqrt{a_2^2 - w^2}$ with $a_1 = 1/(2 - \sqrt{2/\pi})$ and $a_2 = 1 - a_1$. In fact, we show that (10) holds true for $k$-dimensional maximal symmetric spaces generated by three blocks. To do this, for $k \in \mathbb{N}$, $k \geq 2$ and $a_1, l \in [0, 1]$, set
\[
(11) \quad x^{1,k,l} = \frac{a_1(1, \ldots, 1)}{\sqrt{k}}, \quad x^{2,k,l} = l a_2(c_k, d_k, \ldots, d_k)
\]
and
\[
x^{3,k,l} = (1 - l)a_2(1, 0, \ldots, 0)
\]
where \( a_2 = 1 - a_1 \) and \( c_k, d_k \) are nonnegative numbers such that

\[
\sqrt{(c_k)^2 + (k-1)(d_k)^2} = 1
\]

and

\[
\sqrt{k-1} d_k = w.
\]

Here \( w \in [0, 1] \) is a fixed number independent of \( k \). Notice that for any \( a_1 \in [0, 1] \)

\[
\|x_1^{1,k,l}\|_e + \|x_2^{1,k,l}\|_e + \|x_1^{2,k,l}\|_e = a_1 + a_2 = 1,
\]

which shows that the above vectors can be used to estimate from below the function \( f \) from Theorem 2.1.

We start with

**Lemma 3.1.** For any \( k \in \mathbb{N} \) and \( a, b \in \mathbb{R}_+ \)

\[
\sum_{\epsilon \in \{-1,1\}^k} |\langle \epsilon, (a, b, \ldots, b) \rangle| \geq 2^k a
\]

and

\[
\sum_{\epsilon \in \{-1,1\}^k} |\langle \epsilon, (a, a, b, \ldots, b) \rangle| \geq 2^k a,
\]

where for \( x = (x_1, \ldots, x_k) \in \mathbb{R}^k \) and \( y = (y_1, \ldots, y_k) \in \mathbb{R}^k \)

\[
\langle x, y \rangle = \sum_{j=1}^k x_j y_j.
\]

**Proof.** Notice that

\[
\sum_{\epsilon \in \{-1,1\}^k} |\langle \epsilon, (a, b, \ldots, b) \rangle| = 2 \cdot \sum_{\epsilon \in \{-1,1\}^{k-1}} |a + \langle \epsilon, (b, \ldots, b) \rangle| = 2 \cdot \sum_{\epsilon \in \{-1,1\}^{k-2}} |a + b + \langle \epsilon, (b, \ldots, b) \rangle| + |a - b - \langle \epsilon, (b, \ldots, b) \rangle| \geq 2 \cdot \sum_{\epsilon \in \{-1,1\}^{k-2}} |2a| = 2^k a.
\]

The second inequality can be proved in the same way. \( \square \)

**Lemma 3.2.** Let \( f_k \) be the function defined in Th. 2.1 by (9) for \( N = 3 \) and \( k \geq 2 \). Then for any \( a_1 \in [0, 1] \)

\[
\frac{f_k(x_1^{1,k,l}, x_2^{2,k,l}, x_3^{3,k,l})}{2^k(k-1)!} \geq g_{a_1,k,l}(w) := \frac{a_1^2 C_k}{2^{k-1}} + 2\sqrt{k} a_1 a_2 (c_k + 2\sqrt{k} a_1 a_2 (1 - l) + 2a_2^2 l (1 - l) (c_k + (k - 1)d_k) + (a_2 l)^2 (c_k^2 + (k - 1)c_k d_k)}
\]
where

\[ C_k = \sum_{l=0}^{(k-1)/2} \binom{k}{l} (k - 2l) \]

for \( k \) odd,

\[ C_k = \sum_{l=0}^{k/2-1} \binom{k}{l} (k - 2l) \]

for \( k \) even and \( a_2 = 1 - a_1 \).

**Proof.** By (9) and (8),

\[ f_{x_{1},x_{2},x_{3}}^{k}(x_{1},x_{2},x_{3}) = 2^{k} (k-1)! \sum_{i,j=1}^{3} x_{x_{i},k,l}[x_{x_{j},k,l}]^{2} \]

Note that by elementary calculations (compare with [1], Th. 2.8)

\[ x_{x_{1},k,l}[x_{x_{1},k,l}] = (a_{2}^2 / k) 2k! C_k. \]

Also

\[ x_{x_{2},k,l}[x_{x_{2},k,l}] \geq (a_{2} l)^2 2^k (k-1)! c_k (c_k + (k-1) d_k), \]

\[ x_{x_{3},k,l}[x_{x_{3},k,l}] = 2^k (k-1)! a_{2}^2 (1-l)^2, \]

\[ x_{x_{1},k,l}[x_{x_{2},k,l}] = (a_{1} a_{2} l)^2 2^k k! c_k / \sqrt{k}, \]

\[ x_{x_{1},k,l}[x_{x_{3},k,l}] = (a_{1} a_{2} (1-l))^2 k! / \sqrt{k} \]

and

\[ x_{x_{2},k,l}[x_{x_{3},k,l}] = a_{2}^2 (1-l) l (c_k + (k-1) d_k) 2^k (k-1)!. \]

To prove (16), notice that by Lemma 3.1

\[ x_{x_{2},k,l}[x_{x_{2},k,l}] = (a_{2} l)^2 (k-1)! \left( \sum_{\epsilon \in \{-1,1\}^k} \left| \langle \epsilon, (c_k^2, d_k^2, \ldots, d_k^2) \rangle \right| \right. \]

\[ + (k-1) \sum_{\epsilon \in \{-1,1\}^k} \left| \langle \epsilon, (c_k d_k, c_k d_k, d_k^2, \ldots, d_k^2) \rangle \right| \]

\[ \geq (a_{2} l)^2 2^k (k-1)! (c_k^2 + (k-1) c_k d_k), \]

as required. The proof of equalities (17)–(20) follows by elementary calculations. Applying (14)–(20), we get the result. \( \square \)
Lemma 3.3. Let \( g : [0, 1]^3 \to \mathbb{R} \) be defined by
\[
(21) \quad g(a_1, w, l) = a_1^2 \sqrt{2/\pi} + 2a_1a_2l\sqrt{1-w^2}
+ 2a_2a_1(1-l) + (a_2l)^2 w\sqrt{1-w^2} + 2a_2w(1-l)l,
\]
where \( a_2 = 1 - a_1 \). Let, for \( k \in \mathbb{N}, k \geq 2 \), \( x^{1,k,l}, x^{2,k,l}, x^{3,k,l} \in \mathbb{R}^k \) be the vectors associated with \( w \) by (11)–(13). Let \( f_k \) be as in Lemma 3.2. Then for any \((a_1, l, w) \in [0, 1]^3\)
\[
\liminf_k \frac{f_k(x^{1,k,l}, x^{2,k,l}, x^{3,k,l})}{\sqrt{k}2^k(k-1)!} \geq g(a_1, w, l).
\]

Proof. By Lemma 3.2 for any \((l, w) \in [0, 1]^2\)
\[
\frac{f_k(x^{1,k,l}, x^{2,k,l}, x^{3,k,l})}{\sqrt{k}2^k(k-1)!} \geq g_{a_1,k,l}(w)/\sqrt{k}
\]
\[
= \frac{a_1^2C_k}{\sqrt{k}2^{k-1}} + 2a_1a_2l\sqrt{1-w^2} + 2a_2a_1(1-l)
+ 2a_2^2l(1-l)\sqrt{k-lw} + (a_2^2l^2 \sqrt{k-lw} \sqrt{1-w^2})/\sqrt{k}.
\]
By \([1]\), Lemma 2.3, \( \frac{C_k}{2^{k-1}} = \lambda(l^{(k)}_2) \). By \([7]\), \( \lim_k \frac{C_k}{2^{k-1}2^k} = \sqrt{\frac{2}{\pi}} \). Consequently,
\[
\liminf_k \frac{f_k(x^{1,k,l}, x^{2,k,l}, x^{3,k,l})}{\sqrt{k}2^k(k-1)!} \geq \lim_k g_{a_1,k,l}(w)/\sqrt{k} = g(a_1, w, l),
\]
as required. \qed

Remark 3.1. Notice that for any \( a_1 \in [0, 1] \)
\[
g(a_1, w, 1) = a_1^2 \sqrt{2/\pi} + 2a_1a_2\sqrt{1-w^2} + a_2^2 w\sqrt{1-w^2}.
\]
Set \( a_1 = \frac{1}{2-\sqrt{2/\pi}} \). Changing variables from \( w \in [0, 1] \) to \( a_2w \in [0, a_2] \) we get
\[
\max_{w \in [0, 1]} g(a_1, w, 1) = \max_{w \in [0, a_2]} h(w),
\]
where
\[
h(w) = a_1^2 \sqrt{2/\pi} + 2a_1\sqrt{a_2^2 - w^2} + w\sqrt{a_2^2 - w^2}.
\]
In \([3]\), Lemma 5.2 it was shown that
\[
\lim_k \frac{g_{a_1,k,1}(w)}{\sqrt{k}} \geq g(a_1, w, 1) = h(a_2w)
\]
for any \( w \in [0, 1] \). Moreover it can be shown by elementary calculations that for \( a_1 = \frac{1}{2 - \sqrt{2}/\pi} \) the function \( h(w) \) attains its global maximum on \([0, a_2]\) at
\[
  w_0 = \frac{\sqrt{a_1^2 + 2a_2^2} - a_1}{2}.
\]

**Lemma 3.4.** For any \( a_1 \in [0, 1] \) and \( w \in (0, 1] \) there exists \( l \in (0, 1) \) such that \( g(a_1, w, l) > g(a_1, w, 1) \).

**Proof.** First notice that for any \( l \in [0, 1] \) and \( k \in \mathbb{N} \),
\[
  2a_1a_2lc_k + 2a_1a_2(1 - l) \geq 2a_1a_2c_k = 2a_1a_2\sqrt{1 - w^2}.
\]
To end our proof let us consider (for fixed \( a_1 \in [0, 1] \) and \( w \in (0, 1] \)) the function
\[
  u(l) = 2a_2^2w(1 - l)l + (a_2l)^2w\sqrt{1 - w^2}.
\]
Notice that
\[
  u(l) = 2a_2wl + l^2a_2^2w(\sqrt{1 - w^2} - 2).
\]
It is easy to see that \( u'(l) = 0 \) if and only if \( l = l_w = \frac{1}{2 - \sqrt{2}/\pi} \) and that \( u \) attains its global maximum on \([0, 1] \) at \( l_w \in (0, 1) \). By (21) and the above reasoning for any \( a_1 \in [0, 1] \) and \( w \in (0, 1] \), \( g(a_1, w, l_w) > g(a_1, w, 1) \), which shows our claim.

Now we can state the main result of this note

**Theorem 3.1.** For each \( k \in \mathbb{N} \) there exist \( y^{1,k}, y^{2,k}, y^{3,k} \in \mathbb{R}^k \) such that the symmetric spaces \( V^k \) generated by \( y^{1,k}, y^{2,k}, y^{3,k} \) satisfy
\[
  \liminf_k \left( \frac{\lambda_k}{\sqrt{k}} \right) \geq \liminf_k \left( \frac{\lambda(V^k, l_1)}{\sqrt{k}} \right) \geq \max_{(a, w, l) \in [0, 1]^3} g(a, w, l) > \frac{1}{2 - \sqrt{2}/\pi}.
\]

**Proof.** We apply Th. 2.1. Let \( a_1 = \frac{1}{2 - \sqrt{2}/\pi} \) and \( a_2 = 1 - a_1 \). Fix \( k \in \mathbb{N} \), \( N = 3 \), \( w_0 = \frac{\sqrt{(a_1/a_2)^2 + 2 - a_1/a_2}}{2} \). Let \( l_0 = \frac{1}{2 - \sqrt{1 - w_0^2}} \). Let \( y^{1,k}, y^{2,k} \) and \( y^{3,k} \) be the vectors maximizing the function \( f = f^k \) defined by (9). Let \( V^k \) be the symmetric space generated by \( y^{1,k}, y^{2,k} \) and \( y^{3,k} \). By Th. 2.1 and (1),
\[
  \lambda_k^S \geq \lambda(V^k, l_1) = \frac{f^k(y^{1,k}, y^{2,k}, y^{3,k})}{2^k(k - 1)!} \geq \frac{f^k(x^{1,k,l_0}, x^{2,k,l_0}, x^{3,k,l_0})}{2^k(k - 1)!},
\]
where \( x^{1,k,l_0}, x^{2,k,l_0} \) and \( x^{3,k,l_0} \) are as in Lemma 3.3. By Lemma 3.2, Lemma 3.3 and Lemma 3.4 we get the result. □
Remark 3.2. Let \( a_1 = \frac{1}{2 - \sqrt{2/\pi}} \) and \( a_2 = 1 - a_1 \). Lemma 3.3 provides the lower estimate
\[
\liminf_k \left( \frac{\lambda(V^k, l_1)}{\sqrt{k}} \right) \geq g(a_1, w_0, l_0) = 0.83345 \ldots
\]
where \( w_0 = \sqrt{\frac{a_1^2 + 2 - a_1^2}{a_2}} \) and \( l_0 = \frac{a_2 w_0 + a_1 (\sqrt{1-w_0^2} - 1)}{a_2^2 w_0 (2 - 1/w_0^2)} \).

At the end of this note we show how to maximize the function \( g(a, w, l) \) numerically, which will improve the numerical estimate from Remark (3.2).

Lemma 3.5. Let \( g \) be as in Lemma 3.3. Then
\[
\max\{g(a, w, l) : (a, w, l) \in [0, 1]^3\} = \max_{x \in [0, \pi/2]} l(x),
\]
where
\[
l(x) = \sqrt{2/\pi} a(x)^2 + \frac{4}{3} a(x) (1 - a(x)) \left( \cos(x) + \frac{1}{2} \right) + \frac{4}{9} (1 - a(x))^2 \sin(x)(\cos(x) + 1).
\]
with
\[
a(x) = \frac{\sin(x)^2 - \cos(x)^2 - \cos(x)}{\sin(x)^2 - \cos(x)^2 - 3 \sin(x) - \cos(x)}.
\]

Proof. Setting \( w = \sin(x) \), we obtain
\[
g(a, w, l) = a^2 \sqrt{2/\pi} + 2 a(1-a) l \cos(x) + 2(1-a) a(1-l)
+ (1-a)^2 l^2 \sin(x) \cos(x) + 2(1-a)^2 (1-l) \sin(x),
\]
which needs to be maximized on \( 0 \leq l, a \leq 1 \) and \( 0 \leq x \leq \frac{\pi}{2} \). Taking partial derivatives with respect to \( x \) and \( l \) and setting them to zero, after elementary but tedious calculations, one can obtain \( l = \frac{2}{3} \). Hence to maximize \( g \), it is enough to maximize
\[
z(w, x) = a^2 \sqrt{2/\pi} + \frac{4}{3} a(1-a) \cos(x) + \frac{2}{3} (1-a) a
+ \frac{4}{9} (1-a)^2 \sin(x) \cos(x) + \frac{4}{9} (1-a)^2 \sin(x),
\]
on \( 0 \leq a \leq 1 \) and \( 0 \leq x \leq \frac{\pi}{2} \). Taking the partial derivative of \( z \) with respect to \( x \) we easily get that
\[
a = a(x) = \frac{\cos(2x) + \cos(x)}{\cos(2x) + \cos(x) - 3 \sin(x)}.
\]
Hence our problem reduces to maximizing the function $l(x)$ on $[0, \pi/2]$, which proves our lemma.

**Remark 3.3.** Maximizing numerically $l(x)$ on $[0, \pi/2]$ one can get

$$\max \{l(x) : x \in [0,\pi/2]\} = 0.8337894\ldots$$

Hence by Theorem 3.1

$$\liminf_k \left( \frac{\lambda_k^S}{\sqrt{k}} \right) \geq \max_{(a,w,l)\in[0,1]^3} g(a, w, l) = 0.833789\ldots$$

**References**


*Received November 26, 2013*

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