Sergio A. CELANI and Daniela MONTANGIE

HILBERT ALGEBRAS WITH A NECESSITY MODAL OPERATOR

A b s t r a c t. We introduce the variety of Hilbert algebras with a modal operator □, called $H□$-algebras. The variety of $H□$-algebras is the algebraic counterpart of the $\{\to, □\}$-fragment of the intuitionistic modal logic $\textbf{IntK}_{□}$. We will study the theory of representation and we will give a topological duality for the variety of $H□$-algebras. We are going to use these results to prove that the basic implicative modal logic $\textbf{IntK}_{□}$ and some axiomatic extensions are canonical. We shall also to determine the simple and subdirectly irreducible algebras in some subvarieties of $H□$-algebras.

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1. Introduction

We understand by an intuitionistic modal logic any subset of formulas in a propositional language $\mathcal{L}_m$ endowed with a set of unary modal operators $M$ containing all the theorems of intuitionistic propositional logic $\textbf{Int}$, and closed under the rules of Modus Ponens, substitution and the regularity rule $\phi \rightarrow \alpha/m\phi \rightarrow m\alpha$, for each unary operator $m \in M$. In the literature exist several intuitionistic modal logics. There are logics with a necessity modal operator $\Box$, as the basic intuitionistic modal logic $\textbf{IntK}$ (see [19] or [26]). Extensions of $\textbf{IntK}$ was studied in [16], [19], [20], and [22]. Also we have a basic intuitionistic modal logic $\textbf{IntK}_\Diamond$ in the language $\mathcal{L}_\Diamond$, and defined as the smallest logic to contains the axioms $\Diamond(p \lor q) \leftrightarrow \Diamond p \lor \Diamond q$ and $\neg \Diamond \bot$. Extensions of $\textbf{IntK}_\Diamond$ was studied in [12], [19], [20], and [26]. We can also define a logic $\textbf{IntK}_{\Box\Diamond}$, with the modal operators $\Box$ and $\Diamond$, as the smallest logic in the language $\mathcal{L}_{\Box\Diamond}$ containing both $\textbf{IntK}_\Box$ and $\textbf{IntK}_\Diamond$. Extensions of $\textbf{IntK}_{\Box\Diamond}$ was studied in [1], [2], [14], [13], [19], and [20]. Just as Heyting algebras are the algebraic counterpart of $\textbf{Int}$, Heyting algebras with modal operators are the algebraic counterpart of the intuitionistic modal logics $\textbf{IntK}_\Box$, $\textbf{IntK}_\Diamond$ and $\textbf{IntK}_{\Box\Diamond}$.

It is known that the variety $\textbf{Hil}$ of Hilbert algebras is the algebraic semantic of the positive implicative fragment $\textbf{Int}^\rightarrow$ of the intuitionistic propositional calculus $\textbf{Int}$ (see [11], [18] or [24]). So, it is natural to ask for the implicative reducts of some intuitionistic modal logics. Again here we have multiple possibilities. For example, we can studied the fragments $\{\rightarrow, \Box\}$ and $\{\rightarrow, \lor, \Diamond\}$ of the intuitionistic modal logics $\textbf{IntK}_\Box$ and $\textbf{IntK}_\Diamond$, respectively. Another interesting possibility is to study some $\{\rightarrow, \lor, \Box, \Diamond\}$-fragments of $\textbf{IntK}_{\Box\Diamond}$, or the intuitionistic modal logic $\textbf{FS}_{\Box\Diamond}$ defined by Fischer-Servi in [14]. In this paper we will start studying the algebraic semantic of the $\{\rightarrow, \Box\}$-fragment of the intuitionistic normal modal logic $\textbf{IntK}_\Box$. This fragment is denoted by $\textbf{IntK}_\Box^\rightarrow$. The class of algebras associate with $\textbf{IntK}_\Box^\rightarrow$ is the variety $\textbf{Hil}_\Box$ of Hilbert algebras with a necessity modal operator $\Box$. We note that the variety of modal Tarski algebras studied in [5] is the algebraic semantics of the $\{\rightarrow, \Box\}$-fragment of the classical modal logic $\textbf{K}$, and thus is a subvariety of $\textbf{Hil}_\Box$.

The paper is organized as follows. In Section 2 we will recall the definitions and some basic properties of Hilbert algebras and we will recall the topological representation and duality for Hilbert algebras developed in [9].
Also, we will recall the relational semantic of the implicational fragment of intuitionistic logic defined by R. Kirk in [21]. In Section 3 we will introduce the Hilbert algebras with a unary operator □, or $H□$-algebras for short. We will develop the topological representation and duality for $H□$-algebras using the simplified representation given in [9]. In Section 4 we shall characterize the $H□$-algebras that satisfy certain equations by means of first-order conditions defined in the dual space. Each of these varieties corresponds to an axiomatic extension of $\text{IntK}$. In Section 5 we will show that some implicational modal logics are canonical. Finally, in Section 6, we shall determine the simple and subdirectly irreducible algebras of some varieties of $H□$-algebras.

2. Preliminaries

In this section we will fix the terminology adopted in this paper.

**Definition 2.1.** [11] A Hilbert algebra is an algebra $A = \langle A, \rightarrow, 1 \rangle$ of type $(2, 0)$ such that the following axioms hold in $A$:

1. $a \rightarrow a = 1$,
2. $1 \rightarrow a = a$,
3. $a \rightarrow (b \rightarrow c) = (a \rightarrow b) \rightarrow (a \rightarrow c)$,
4. $(a \rightarrow b) \rightarrow ((b \rightarrow a) \rightarrow a) = (b \rightarrow a) \rightarrow ((a \rightarrow b) \rightarrow b)$.

The variety of Hilbert algebras is denoted by Hil. It is easy to see that the binary relation $\leq$ defined in a Hilbert algebra $A$ by $a \leq b$ if and only if $a \rightarrow b = 1$ is a partial order on $A$ with greatest element 1.

Given a Hilbert algebra $A$ and a sequence $a, a_1, \ldots, a_n \in A$, we define:

$$(a_1, \ldots, a_n; a) = \begin{cases} a_1 \rightarrow a & \text{if } n = 1, \\ a_1 \rightarrow (a_2, \ldots, a_n; a) & \text{if } n > 1. \end{cases}$$

A subset $F \subseteq A$ is an *implicative filter* or *deductive system* of $A$ if $1 \in F$, and if $a, a \rightarrow b \in F$ then $b \in F$. The set of all implicative filters of a Hilbert algebra $A$ is denoted by $\text{Fi}(A)$. The implicative filter generated
by a set $X$ is $\langle X \rangle = \bigcap \{ F \in \text{Fi}(A) : X \subseteq F \}$.
If $X = \{a\}$, then we write $\langle a \rangle = \{ b \in A : a \leq b \}$. The implicative filter generated by a subset $X \subseteq A$
can be characterized as the set $\langle X \rangle = \{ a \in A : \exists \{ a_1, \ldots, a_n \} \subseteq X : (a_1, \ldots, a_n; a) = 1 \}$.

Let $F \in \text{Fi}(A) - \{A\}$. We will say that $F$ is irreducible if and only if for any $F_1, F_2 \in \text{Fi}(A)$ such that $F = F_1 \cap F_2$, it follows that $F = F_1$ or $F = F_2$. The set of all irreducible implicative filters of a Hilbert algebra $A$
is denoted by $X(A)$. Let us recall that an implicative filter $F$ is irreducible if for every $a, b \in A$ such that $a, b \notin F$ there exists $c \notin F$ such that $a, b \leq c$
(see [4], [11] or [24]). A subset $I$ of $A$ is called an order-ideal of $A$ if $b \in I$ and $a \leq b$, then $a \in I$, and for each $a, b \in I$ there exists $c \in I$ such that $a \leq c$ and $b \leq c$. The set of all order-ideals of $A$ will be denoted by $\text{Id}(A)$.

The following is a Hilbert algebra analogue of Birkhoff's Prime Filter Lemma and it is proved in [6]. We note that in [21] is used a similar theorem (see also [27]), but with the notion of $a$-maximal filter. It is not difficult to check that every $a$-maximal filter is irreducible, but the converse is not generally valid.

**Theorem 2.2.** Let $A$ be a Hilbert algebra. Let $F \in \text{Fi}(A)$ and let $I \in \text{Id}(A)$ such that $F \cap I = \emptyset$. Then, there exists $x \in X(A)$ such that $F \subseteq x$ and $x \cap I = \emptyset$.

A bounded Hilbert algebra is a Hilbert algebra $A$ with an element $0 \in A$ such that $0 \rightarrow a = 1$, for every $a \in A$. The notation $\neg a$ means $a \rightarrow 0$. The variety of bounded Hilbert algebras is denoted by $\text{Hil}^0$.

**Lemma 2.3.** Let $A \in \text{Hil}^0$. Then,

1. If $a \in x$, then $\neg a \notin x$, for every $x \in X(A)$.
2. If $\neg a \notin y$ then there exists $x \in X(A)$ such that $y \subseteq x$ and $a \in x$, for all $y \in X(A)$.

**Proof.** (1) Suppose that $\neg a \in x$. So, $a \rightarrow 0 \in x$. As $a \in x$, we get that $0 \in x$, which is impossible because $x$ is a proper implicative filter. (2) This is an immediate consequence of Theorem 2.2. \qed

For a partially ordered set $\langle X, \leq \rangle$ and $Y \subseteq X$, let $[Y] = \{ x \in X : \exists y \in Y : y \leq x \}$
and

\[(Y) = \{x \in X : \exists y \in Y : x \leq y\}.\]

If \(Y\) is the singleton \(\{y\}\), then we write \([y]\) and \((y)\) instead of \([\{y\}]\) and \((\{y\})\), respectively. We call \(Y\) an upset (resp. downset) if \(Y = [Y]\) (resp. \(Y = (Y)\)). The set of all upset subsets of \(X\) is denoted by \(\text{Up}(X)\). It is known that \(\langle \text{Up}(X), \Rightarrow_{\leq}, X \rangle\) is a Hilbert algebra where the implication \(\Rightarrow_{\leq}\) is defined by

\[U \Rightarrow_{\leq} V = (U \cap V)^c = \{x : [x] \cap U \subseteq V\}\]

for \(U, V \in \text{Up}(X)\).

An \(H\)-set or expanded Kripke frame (in the terminology of Kirk in [21]) is a triple \(\langle X, \leq, K \rangle\) where \(\langle X, \leq \rangle\) is a poset and \(\emptyset \neq K \subseteq \mathcal{P}(X)\). Every \(H\)-set defines a structure \(H_K(X)\) as follows:

\[H_K(X) = \{U \in \mathcal{P}(X) : \exists W \in K \text{ and } \exists V \subseteq W \text{ such that } (U = W \Rightarrow_{\leq} V)\}.\]

As is proved in [21] and [7] the triple \(H_K(X) = \langle H_K(X), \Rightarrow_{\leq}, X \rangle\) is a Hilbert algebra and a subalgebra of \(\langle \text{Up}(X), \Rightarrow_{\leq}, X \rangle\). The algebra \(H_K(X)\) is called the dual Hilbert algebra of \(\langle X, \leq, K \rangle\).

Consider a pair \(\langle X, K \rangle\) where \(X\) is a set and \(\emptyset \neq K \subseteq \mathcal{P}(X)\). We define a relation \(\leq_K \subseteq X \times X\) by

\[x \leq_K y \text{ iff } \forall W \in K(x \notin W \text{ then } y \notin W).\]

It is easy to see that \(\leq_K\) is a reflexive and transitive relation. For each \(Y \subseteq X\), let

\[\text{sat}(Y) = \bigcap \{W : Y \subseteq W \text{ and } W \in K\}\]

and

\[\text{cl}(Y) = \bigcap \{X - W : Y \cap W = \emptyset \text{ and } W \in K\}.\]

When \(K\) is a basis of a topology \(\mathcal{T}\) defined on \(X\), the relation \(\leq_K\) is the specialization dual order of \(X\), \(\text{sat}(Y)\) is the saturation of \(Y\), and \(\text{cl}(Y)\) is the closure of \(Y\). We note that \(\leq_K\) can be defined in terms of the operator \(\text{cl}\) as follows: \(x \leq_K y\) iff \(y \in \text{cl}([x]) = \text{cl}(x)\). If \(X\) is \(T_0\) then the relation \(\leq_K\) is a partial order. Moreover, if \(X\) is \(T_0\) then \(\text{cl}(Y) = [Y]_{\leq_K}\), \(\text{sat}(Y) = (Y)_{\leq_K}\), and every open (resp. closed) subset is a downset (resp. upset) respect to \(\leq_K\).
Let $X$ be a topological space. We recall that a subset $Y \subseteq X$ is irreducible provided for any closed subsets $Y_1$ and $Y_2$, if $Y = Y_1 \cup Y_2$ then $Y = Y_1$ or $Y = Y_2$. A topological space $X$ is sober if, for every irreducible closed set $Y$, there exists a unique $x \in X$ such that $\text{cl}(x) = Y$. Notice that a sober space is automatically $T_0$. A topological space $\langle X, T \rangle$ with a base $\mathcal{K}$ we will denote by $\langle X, T_{\mathcal{K}} \rangle$ or simply by $\langle X, \mathcal{K} \rangle$. Recall that the relation $\leq_{\mathcal{K}}$ defined in (3) is an order when the space is $T_0$. From now on, for every sober topological space $\langle X, \mathcal{K} \rangle$ we shall write $\leq$ instead of $\leq_{\mathcal{K}}$.

**Definition 2.4.** \cite{9} A Hilbert space or $H$-space is a topological space $\langle X, \mathcal{K} \rangle$ such that:

H1. $\mathcal{K}$ is a base of open and compact subsets for the topology $T_{\mathcal{K}}$ on $X$,

H2. For every $A, B \in \mathcal{K}$, $\text{sat}(A \cap B^c) \in \mathcal{K}$,

H3. $\langle X, \mathcal{K} \rangle$ is sober.

Let $A$ be a Hilbert algebra. Let us consider the poset $\langle X(A), \subseteq \rangle$ and the mapping $\varphi : X(A) \to \text{Up}(X(A))$ defined by

$$\varphi(a) = \{ x \in X(A) : a \in x \}.$$  

In \cite{8} it was proved that the family $\mathcal{K}_A = \{ \varphi(a)^c : a \in A \}$ is a basis for a topology $T_{\mathcal{K}_A}$ and the pair $\langle X(A), \mathcal{K}_A \rangle$ is an $H$-space, called the dual space of $A$. If $A$ is a bounded Hilbert algebra, then $\varphi(0) = \emptyset$. So, $X(A) = \varphi(0)^c \in \mathcal{K}_A$ and consequently the $H$-space $\langle X(A), \mathcal{K}_A \rangle$ is compact.

If $\langle X, \mathcal{K} \rangle$ is an $H$-space, then for each $x \in X$, the set

$$\varepsilon(x) = \{ U \in D(X) : x \in U \}$$

belongs to $X(D(X))$, where $D(X) = \{ U : U^c \in \mathcal{K} \}$. Thus, the mapping $\varepsilon : X \to X(D(X))$ is well-defined and it is an homeomorphism between the topological spaces $\langle X, \mathcal{K} \rangle$ and $\langle X(D(X)), \mathcal{K}_{D(X)} \rangle$.

Let $A$ and $B$ be Hilbert algebras. A mapping $h : A \to B$ is a semi-homomorphism if $h(1) = 1$, and $h(a \to b) \leq h(a) \to h(b)$, for all $a, b \in A$. A mapping $h : A \to B$ is a homomorphism if $h$ is a semi-homomorphism such that $h(a) \to h(b) \leq h(a \to b)$, for all $a, b \in A$. Note that a semi-homomorphism is a monotone map.
Lemma 2.5. Let $A$ and $B$ be Hilbert algebras. Let $h : A \to B$ be a semi-homomorphism. If $x \in X(A)$, then $(h(x^c)) \in \text{Id}(B)$.

Proof. Assume that $x \in X(A)$. Let $a, b \in (h(x^c))$. Then there exist $c, d \notin x$ such that $a \leq h(c)$ and $b \leq h(d)$. Since $x$ is irreducible, there exists $e \notin x$ such that $c, d \leq e$, and as $h$ is monotonic, $a \leq h(e)$ and $b \leq h(e)$. So, $h(e) \in (h(x^c))$, and thus $(h(x^c))$ is an order-ideal. □

We denote by $\text{Hil}_S$ the category of $H$-algebras and semi-homomorphisms between Hilbert algebras. Similarly, we denote by $\text{Hil}_H$ the category of $H$-algebras and homomorphisms. Clearly, $\text{Hil}_H$ is a subcategory at $\text{Hil}_S$.

Definition 2.6. Let $\langle X_1, K_1 \rangle$ and $\langle X_2, K_2 \rangle$ be $H$-spaces. Let us consider a relation $R \subseteq X_1 \times X_2$. We say that $R$ is an $H$-relation if $R^{-1}(U) \in K_1$, for every $U \in K_2$, and $R(x)$ is a closed subset of $X_2$, for all $x \in X_1$.

An $H$-relation $R \subseteq X_1 \times X_2$ is an $H$-functional relation if for each pair $(x, y) \in R$, there exists $z \in X_1$ such that $x \leq z$ and $R(z) = [y]$.

$\mathcal{SR}$ ($\mathcal{SRF}$) denote the category whose objects are $H$-spaces and whose morphisms are $H$-relations ($H$-functional relations). By Theorem 3.5 and Theorem 3.7 in [8] we have that the categories $\mathcal{SR}$ ($\mathcal{SRF}$) and $\text{Hil}_S$ ($\text{Hil}_H$) are dually equivalents.

3. $H\Box$-algebras: representation and duality

In this section we shall define the Hilbert algebras with a modal operator of necessity $\Box$.

Definition 3.1. A Hilbert algebra with a modal operator $\Box$, or $H\Box$-algebra for short, is a pair $A = \langle A, \Box \rangle$ where $A$ is a Hilbert algebra and $\Box$ is a semi-homomorphism defined on $A$, i.e., $\Box 1 = 1$, and $\Box(a \to b) \leq \Box a \to \Box b$, for all $a, b \in A$.

We denote by $\text{Hil}_\Box$ the variety of $H\Box$-algebras. The variety $\text{Hil}_\Box$ correspond to the $\{\Box, \to\}$-reduct of the variety of Heyting algebras with a modal operator $\Box$ (see, for example [10]). Moreover, the variety of Tarski modal algebras introduced in [5] is a subvariety of $\text{Hil}_\Box$.

Let $A, B \in \text{Hil}_\Box$. A map $h : A \to B$ is a $\Box$-semi-homomorphism ($\Box$-homomorphism) if $h$ is a semi-homomorphism (homomorphism) such
that \( h(\Box a) = \Box(h(a)) \), for all \( a \in A \). We denote by \( \text{Hil}_{\Box}S \) the category of \( H\Box \)-algebras with \( \Box \)-semi-homomorphisms and by \( \text{Hil}_{\Box}H \) the category of \( H\Box \)-algebras with \( \Box \)-homomorphisms.

Let \( X \) be a set and \( Q \) a binary relation defined on \( X \). For each \( U \in \mathcal{P}(X) \) consider the set
\[
\Box_Q(U) = \{ x \in X : Q(x) \subseteq U \}.
\]

**Example 3.2.** [19] An intuitionistic modal Kripke frame is a relational structure \( \mathcal{F} = (X, \leq, Q) \), where \( (X, \leq) \) is a poset, and \( Q \) is a binary relation defined on \( X \) such that \( \leq \circ Q \subseteq Q \circ \leq \), where \( \circ \) is the composition of relations. It is easy to see that \( \langle \up(U(X), \Rightarrow, \cap, \cup, \Box_Q, \emptyset, X \rangle \) is a Heyting algebra with a modal operator \( \Box \). Thus, \( \langle \up(U(X), \Rightarrow, \leq, \Box_Q, X \rangle \in \text{Hil}_{\Box} \).

**Definition 3.3.** A triple \( \langle X, \mathcal{K}, Q \rangle \) is an \( H\Box \)-frame if \( (X, \leq) \) is a poset and \( (\leq \circ Q) \subseteq (Q \circ \leq) \), where \( \leq \) is \( \leq_{\mathcal{K}} \).

An \( H\Box \)-frame \( \langle X, \mathcal{K}, Q \rangle \) is a general \( H\Box \)-frame if:

1. sat\((U \cap V^c) \in \mathcal{K} \), for every \( U, V \in \mathcal{K} \).
2. \( Q^{-1}(U) \in \mathcal{K} \), for every \( U \in \mathcal{K} \).

**Lemma 3.4.** If \( \mathcal{F} = (X, \mathcal{K}, Q) \) is a general \( H\Box \)-frame, then
\[
A(\mathcal{F}) = \langle \up(X), \Rightarrow, \Box_Q, X \rangle \in \text{Hil}_{\Box}.
\]
and \( \langle D(X), \Box_Q \rangle \) is a subalgebra of \( A(\mathcal{F}) \).

**Proof.** As \( (X, \leq) \) is a poset, we have that \( \langle \up(X), \Rightarrow, X \rangle \) is a Hilbert algebra. We note that \( \Box_Q(U) \in \up(X) \), for every \( U \in \up(X) \), because \( (\leq \circ Q) \subseteq (Q \circ \leq) \). Moreover, as \( \Box_Q(U) = Q^{-1}(U^c)^c \) we get that \( \Box_Q(U) \in D(X) \), because \( Q^{-1}(U^c) \in \mathcal{K} \) for every \( U \in D(X) \). Finally, it is immediate to see that \( \langle D(X), \Rightarrow, \leq_{\mathcal{K}}, X \rangle \) is a subalgebra of the Hilbert algebra \( \langle \up(X), \Rightarrow, \leq_{\mathcal{K}}, X \rangle \).

Let \( A \in \text{Hil}_{\Box} \). For each \( n \geq 0 \), \( n \in \mathbb{N} \), we define inductively the formula \( \Box^n a \) as \( \Box^0 a = a \) and \( \Box^{n+1} a = \Box(\Box^n a) \). Let \( S \) be a subset of \( A \). We define the following sets:
\[
\Box(S) = \{ \Box a \in A : a \in S \} \quad \text{and} \quad \Box^{-1}(S) = \{ a \in A : \Box a \in S \}.
\]
We note that \( \Box^{-1}(F) \in \text{Fi}(A) \), when \( F \in \text{Fi}(A) \). We note also that by Lemma 2.5 \( \Box(x^c) \) is an order-ideal, when \( x \in X(A) \).
Lemma 3.5. Let \( A \in \text{Hil} \). Let \( F \in \text{Fi}(A) \) and \( a \in A \). Then \( \square a \notin F \) iff there exists \( x \in X(A) \) such that \( \square^{-1}(F) \subseteq x \) and \( a \notin x \).

Proof. The proof follows taking into account that \( \square^{-1}(F) \) is an implicative filter and Theorem 2.2.

Let \( A \) be an \( H \square \)-algebra. By the results given in [8], the binary relation \( Q_A \subseteq X(A) \times X(A) \) given by

\[(x, y) \in Q_A \text{ iff } \square^{-1}(x) \subseteq y,\]

for \( x, y \in X(A) \), is the \( H \)-relation associated with the modal operator \( \square \). So, \( Q_A^{-1}(U) \in \mathcal{K}_A \), for every \( U \in \mathcal{K}_A \). It is easy to see that \( Q_A \) satisfies the condition \( Q_A = (\subseteq \circ Q_A) = (Q_A \circ \subseteq) \). Moreover, by Proposition 2.1 in [8] we have that if \( U, V \in \mathcal{K}_A \), then \( \text{sat}(U \cap V^c) \in \mathcal{K}_A \). Thus, the triple

\[ \mathcal{F}(A) = \langle X(A), \mathcal{K}_A, Q_A \rangle, \]

is a general \( H \square \)-frame.

Now we shall define the \( H \square \)-spaces, and we will see that its structures are a particular class of general \( H \square \)-frames.

Definition 3.6. A triple \( \langle X, \mathcal{K}, Q \rangle \) is an \( H \square \)-space if \( \langle X, \mathcal{K} \rangle \) is an \( H \)-space and \( Q \subseteq X \times X \) is an \( H \)-relation.

As \( Q \) is an \( H \)-relation in every \( H \square \)-space \( \langle X, \mathcal{K}, Q \rangle \), by Theorem 3.1.(1) in [8] we get that \( (\subseteq \circ Q) = Q = (Q \circ \subseteq) \) is valid in any \( H \square \)-space. Consequently, we have the following result.

Lemma 3.7. Every \( H \square \)-space is a general \( H \square \)-frame.

Thus, if \( \langle X, \mathcal{K}, Q \rangle \) is an \( H \square \)-space, then \( \langle D(X), Q \rangle \) is an \( H \square \)-algebra.

Theorem 3.8 (of Representation). For each \( H \square \)-algebra \( \langle A, \square \rangle \) there exists an \( H \square \)-space \( \langle X, \mathcal{K}, Q \rangle \) such that \( \langle A, \square \rangle \) is isomorphic to \( \langle D(X), \square Q \rangle \).

Proof. Since \( \langle X(A), \mathcal{K}_A \rangle \) is an \( H \)-space and \( Q_A \) is an \( H \)-relation, we have that \( \langle X(A), \mathcal{K}_A, Q_A \rangle \) is an \( H \square \)-space. By Lemma 3.5, we have that \( \varphi(\square a) = \square Q_A(\varphi(a)) \), for each \( a \in A \). So, \( \langle D(X(A)), \square Q_A \rangle \) is an \( H \square \)-algebra. By Theorem 2.1 in [8] we get that \( \varphi \) is a Hilbert isomorphism. Thus, \( \langle A, \square \rangle \) is isomorphic to \( \langle D(X(A)), \square Q_A \rangle \). \( \Box \)
Definition 3.9. Let $\langle X_1, K_1, Q_1 \rangle$ and $\langle X_2, K_2, Q_2 \rangle$ be $H\Box$-spaces and $R \subseteq X_1 \times X_2$ be an $H$-relation. We say that $R$ is an $H\Box$-relation if $R$ commutes with $Q$, i.e., $Q_1 \circ R = R \circ Q_2$.

If $R \subseteq X_1 \times X_2$ is an $H$-functional relation such that $R$ commutes with $Q$, then $R$ is an $H\Box$-functional relation.

$\mathcal{M_{\Box\Box}}$ denote the category of $H\Box$-spaces and $H\Box$-relations. We will prove that this category is dually equivalent to $\operatorname{Hil}_{\Box\Box}$.

Let $\langle X, K \rangle$ an $H$-space and consider the map $\varepsilon : X \to X \langle D(X) \rangle$ defined by $\varepsilon(x) = \{U \in D(X) : x \in U\}$. By Corollary 3.1 in [8] we get that the relation $\varepsilon^* \subseteq X \times X \langle D(X) \rangle$ given by

$$(x, P) \in \varepsilon^* \iff \varepsilon(x) \subseteq P$$

is an $H$-relation. Now, we will prove that $\varepsilon^*$ is a morphism of $H\Box$-spaces.

Theorem 3.10. Let $\langle X, K, Q \rangle$ an $H\Box$-space. Then, the mapping $\varepsilon$ is an homeomorphism between the $H\Box$-spaces $\langle X, K, Q \rangle$ and $\langle X \langle D(X) \rangle, K_{D(X)}, Q_{D(X)} \rangle$ such that

$$(x, y) \in Q \iff (\varepsilon(x), \varepsilon(y)) \in Q_{D(X)},$$

where $Q_{D(X)}$ is the $H\Box$-relation associated with the modal operator $\Box Q$. Moreover, the relation $\varepsilon^*$ is a morphism of $H\Box$-spaces.

Proof. As $\langle X, K, Q \rangle$ is an $H\Box$-space, $\langle D(X), \Box Q \rangle$ is an $H\Box$-algebra and by Theorem 3.8, the triple $\langle X \langle D(X) \rangle, K_{D(X)}, Q_{D(X)} \rangle$ is an $H\Box$-space where $(F, P) \in Q_{D(X)}$ iff $\Box Q^{-1}(F) \subseteq P$, for all $F, P \in X \langle D(X) \rangle$. By Theorem 2.2 in [8] we get that $\varepsilon$ is an homeomorphism between the $H$-spaces $\langle X, K \rangle$ and $\langle X \langle D(X) \rangle, K_{D(X)} \rangle$, being $K_{D(X)} = \{\varphi(U)^\circ : U \in D(X)\}$.

Let $(x, y) \in Q$. We prove that $(\varepsilon(x), \varepsilon(y)) \in Q_{D(X)}$, i.e., $\Box Q^{-1}(\varepsilon(x)) \subseteq \varepsilon(y)$. Let $U \in D(X)$ such that $U \in \Box Q^{-1}(\varepsilon(x))$. So, $Q(x) \subseteq U$ and as $y \in Q(x)$, we get that $y \in U$. This is, $U \in \varepsilon(y)$. Now, assume that $\Box Q^{-1}(\varepsilon(x)) \subseteq \varepsilon(y)$ and suppose that $(x, y) \notin Q$. As $Q(x)$ is a closed subset of $\langle X, K \rangle$, there exists $U \in D(X)$ such that $Q(x) \subseteq U$ and $y \notin U$. This is, $U \in \Box Q^{-1}(\varepsilon(x))$ and $U \notin \varepsilon(y)$, which contradicts the assumption.

Now, we will prove that $Q \circ \varepsilon^* = \varepsilon^* \circ Q_{D(X)}$. Let $x \in X$ and $P \in X \langle D(X) \rangle$ such that $(x, P) \in Q \circ \varepsilon^*$. So, there exists $y \in X$ such that $(x, y) \in Q$ and $(y, P) \in \varepsilon^*$. This is, $\varepsilon(y) \subseteq P$. As $(x, y) \in Q$, we have
$\langle \varepsilon(x), \varepsilon(y) \rangle \in Q_{D(X)}$, i.e., $\square_Q^{-1}(\varepsilon(x)) \subseteq \varepsilon(y) \subseteq P$. Thus, $\langle \varepsilon(x), P \rangle \in Q_{D(X)}$. It is clear that $\langle x, \varepsilon(x) \rangle \in \varepsilon^*$. So, $\langle x, P \rangle \in \varepsilon^* \circ Q_{D(X)}$. Thus, $Q \circ \varepsilon^* \subseteq \varepsilon^* \circ Q_{D(X)}$. Assume that $(x, P) \in \varepsilon^* \circ Q_{D(X)}$. So, there exists $F \in X(D(X))$ such that $\varepsilon(x) \subseteq F$ and $\square_Q^{-1}(F) \subseteq P$. As $\varepsilon$ is onto, there exists $f, p \in X$ such that $F = \varepsilon(f)$ and $P = \varepsilon(p)$. So, $\square_Q^{-1}(\varepsilon(x)) \subseteq \square_Q^{-1}(\varepsilon(f)) \subseteq \varepsilon(p)$. Then, $(\varepsilon(x), \varepsilon(p)) \in Q_{D(X)}$ and consequently, $(x, p) \in Q$. It is clear that $(p, P) \in \varepsilon^*$. So, $(x, P) \in Q \circ \varepsilon^*$. 

In [8] it was proved that if $\langle X_1, \mathcal{K}_1 \rangle$ and $\langle X_2, \mathcal{K}_2 \rangle$ are $H$-spaces and $R \subseteq X_1 \times X_2$ is an $H$-relation then the mapping $h_R : D(X_2) \to D(X_1)$ defined by

$$h_R(U) = \{ x \in X_1 \mid R(x) \subseteq U \}$$

is a semi-homomorphism.

**Theorem 3.11.** Let $\langle X_1, \mathcal{K}_1, Q_1 \rangle$ and $\langle X_2, \mathcal{K}_2, Q_2 \rangle$ be $H\square$-spaces and $R \subseteq X_1 \times X_2$ be an $H\square$-relation. Then, $h_R$ is a morphism of Hil$\square$S.

**Proof.** We will prove that $h_R(\square_Q(U)) = \square_Q(h_R(U))$, for each $U \in D(X_2)$. Let $x \in X_1$. Then

$$x \in h_R(\square_Q(U)) \iff R(x) \subseteq \square_Q(U) \iff Q_2(R(x)) \subseteq U$$

$$\text{iff } R(Q_1(x)) \subseteq U \iff \forall z \in Q_1(x) \forall z \in Q_1(x) (R(z) \subseteq U)$$

$$\text{iff } Q_1(x) \subseteq h_R(U) \iff x \in \square_Q(h_R(U)).$$

By the above Theorem and Theorem 3.7 in [8], we have the following result.

**Corollary 3.12.** Let $\langle X_1, \mathcal{K}_1, Q_1 \rangle$ and $\langle X_2, \mathcal{K}_2, Q_2 \rangle$ be $H\square$-spaces and $R \subseteq X_1 \times X_2$ be an $H\square$-functional relation. Then, $h_R$ is a morphism of Hil$\square$H.

Let $A, B$ be Hilbert algebras and $h : A \to B$ be a semi-homomorphism. In [8] it was proved that the relation $R_h \subseteq X(B) \times X(A)$ defined by

$$(x, y) \in R_h \text{ iff } h^{-1}(x) \subseteq y$$

is an $H$-relation. Now, we will study $R_h$ when $h$ is a semi-homomorphism defined between $H\square$-algebras that commutes with $\square$. 
Theorem 3.13. Let $A, B \in \text{Hil}_\Box$ and let $h : A \to B$ be a $\Box$-semi-homomorphism. Then, $R_h$ is a morphism of $\mathcal{M}_\Box \mathcal{S} \mathcal{R}$.

Proof. If we prove that $R_h \circ Q_A = Q_B \circ R_h$, the assertion follows. Let $x \in X(B)$ and $y \in X(A)$ such that $(x, y) \in R_h \circ Q_A$. So, there exists $z \in X(A)$ such that $z \in R_h(x)$ and $(z, y) \in Q_A$, i.e., $h^{-1}(x) \subseteq z$ and $\Box^{-1}(z) \subseteq y$. Consider the implicative filter $\Box^{-1}(x)$ and the order-ideal $(h(y^c))$ of $B$. Suppose that there exists $a \in \Box^{-1}(x) \cap (h(y^c))$. So, $\Box a \in x$ and there exists $b \in y^c$ such that $a \leq h(b)$. As $\Box a \leq \Box(h(b)) = h(\Box b)$, we get that $h(\Box b) \in x$. Thus, $\Box b \in z$ and so, $b \in y$, which is a contradiction. Thus, $\Box^{-1}(x) \cap (h(y^c)) = \emptyset$. So, there exists $w \in X(B)$ such that $\Box^{-1}(x) \subseteq w$ and $(h(y^c)) \cap w = \emptyset$. This is, there exists $w \in X(B)$ such that $w \in Q_B(x)$ and $h^{-1}(w) \subseteq y$, i.e., $(w, y) \in R_h$. Therefore, $y \in R_h(Q_B(x))$. Thus, $R_h \circ Q_A \subseteq Q_B \circ R_h$. The proof of the other inclusion is similar.

By Theorem 3.13 and Theorem 3.7 in [8] we have the following result.

Corollary 3.14. Let $A, B \in \text{Hil}_\Box$ and let $h : A \to B$ be a $\Box$-homomorphism. Then $R_h$ is an $H\Box$-functional relation.

From Theorem 3.11, we conclude that the functor $D : \mathcal{M}_\Box \mathcal{S} \mathcal{R} \to \text{Hil}_\Box \mathcal{S}$ defined by

$$D(X) = \langle D(X), \Box_Q \rangle \quad \text{if} \quad \langle X, K, Q \rangle \text{ is an } H\Box\text{-space},$$
$$D(R) = h_R \quad \text{if} \quad R \text{ is an } H\Box\text{-relation.}$$

is a contravariant functor. By Remark 3.1 in [8], Theorem 3.8 and Theorem 3.13, we conclude that the functor $X : \text{Hil}_\Box \mathcal{S} \to \mathcal{M}_\Box \mathcal{S} \mathcal{R}$ defined by

$$X(A) = \langle X(A), K_A, Q_A \rangle \quad \text{if} \quad A \text{ is an } H\Box\text{-algebra},$$
$$X(h) = R_h \quad \text{if} \quad h \text{ is a } \Box\text{-semi-homomorphism}$$

is a contravariant functor. From the Lemmas 3.4 and 3.5 in [8] and Theorems 3.8 and 3.10, we give the following result.

Theorem 3.15. The categories $\text{Hil}_\Box \mathcal{S}$ and $\mathcal{M}_\Box \mathcal{S} \mathcal{R}$ are dually equivalent.

Corollary 3.16. The category $\text{Hil}_\Box \mathcal{H}$ is dually isomorphic to the category of $H\Box$-spaces with $H\Box$-functional relations.
4. Some subvarieties of $H\Box$-algebras

The variety of $H\Box$-algebras generated by a finite set of identities $\Gamma$ will be denoted by $\text{Hil}_{\Box} + \{\Gamma\}$. We shall consider some particular varieties of $H\Box$-algebras. These varieties are the algebraic counterpart of extensions of the implicative fragments of the intuitionistic modal logic $\text{IntK}_{\Box}$. Let us consider the following identities:

- **S** \( a \to \Box a \approx 1 \),
- **S\(_n\)** \( a \to \Box^n a \approx 1 \),
- **T** \( \Box a \to a \approx 1 \),
- **4** \( \Box a \to \Box^2 a \approx 1 \),
- **wD** \( \Box^2 a \to \Box a \approx 1 \),
- **5** \( (\Box a \to \Box b) \to \Box(\Box a \to \Box b) \approx 1 \),
- **6** \( \Box^2 a \to \Box a \approx 1 \).

**Remark 4.1.** It is not hard to prove that $\text{Hil}_{\Box} + \{5\}$ and $\text{Hil}_{\Box} + \{S\}$ are subvarieties of $\text{Hil}_{\Box} + \{4\}$.

Following the standard notation, we shall identify two important sub-varieties of $\text{Hil}_{\Box}$:

- $\text{Hil}_{\Box}S_4 = \text{Hil}_{\Box} + \{T, 4\}$,
- $\text{Hil}_{\Box}S_5 = \text{Hil}_{\Box} + \{T, 5\}$.

It is clear that $\text{Hil}_{\Box}S_5$ is subvariety of $\text{Hil}_{\Box}S_4$. The variety $\text{Hil}_{\Box}S_4$ is a generalization of the topological closure Boolean algebras, and the variety $\text{Hil}_{\Box}S_5$ is a generalization of the monadic Boolean algebras. Similar to the proven in [5], each one of the previous identities are characterized by means of first-order conditions.

Let $Q$ be a binary relation defined on a set $X$. For each $n \geq 0$ we define inductively the relation $Q^n$ as follows: $(x, y) \in Q^0$ iff $x = y$, and $(x, y) \in Q^{n+1} = Q^n \circ Q$, where $\circ$ is the composition of relations. Also we define the binary relation $Q^* = \bigcup \{Q^n : n \geq 0\}$.

The next result is a generalization of Lemma 3.5 applied to irreducible implicative filters.

**Lemma 4.2.** Let $A \in \text{Hil}_{\Box}$ and let $\langle X, K, Q \rangle$ be its dual space. Let $x \in X$ and $a \in A$. For each $n \in \mathbb{N}$, $\Box^n a \notin x$ iff there exists $y \in X$ such that $(x, y) \in Q^n$ and $a \notin y$. 
Proof. The proof is by induction on \( n \). It is immediately for \( n = 0 \). Assume that \( \Box^n a \notin x \) implies that there exists \( y \in X \) such that \( (x, y) \in Q^n \) and \( a \notin y \). Suppose that \( \Box^{n+1} a \notin x \). This is, \( \Box (\Box^n a) \notin x \). By Lemma 3.5, there exists \( y \in X \) such that \( \Box^{-1}(x) \subseteq y \) and \( \Box^n a \notin y \). By assumption, there exists \( z \in X \) such that \( (y, z) \in Q^n \) and \( a \notin z \). Since \( (x, y) \in Q \) and \( (y, z) \in Q^n \), we get that \( (x, z) \in Q^{n+1} \).

Consider that if there exists \( y \in X \) such that \( (x, y) \in Q^n \) and \( a \notin y \), then \( \Box^n a \notin x \). Suppose that \( (x, y) \in Q^{n+1} \) and \( a \notin x \). So, there exists \( z \in X \) such that \( (x, z) \in Q^n \) and \( (z, y) \in Q \). Therefore, \( \Box^{-1}(z) \subseteq y \) and as \( a \notin y \), we have that \( \Box a \notin z \). Thus, \( (x, z) \in Q^n \) and \( \Box a \notin z \). By assumption, \( \Box^{n+1} a \notin x \).

Let \( \langle X, K, Q \rangle \) be an \( H\Box \)-space. Following the notation used in [19], we denote by \( \Phi \) and \( \Phi' \) the next first-order conditions:

\[ \Phi \iff \forall x \forall y [xQy \land yQz \Rightarrow \exists w (x \leq w \land wQz \land \forall v (wQv \Rightarrow yQv))] . \]

\[ \Phi' \iff \forall x \forall y [xQy \land yQz \Rightarrow \exists w (x \leq w \land wQz \land yQw)] . \]

Remark 4.3. Let \( \langle X, K, Q \rangle \) be an \( H\Box \)-space. Note that \( \Phi' \) (or \( \Phi \)) implies the transitivity of \( Q \). In fact. Let \( x, y, z \in X \) such that \( xQy \) and \( yQz \). By \( \Phi' \), there exists \( w \in X \) such that \( x \leq w \), \( wQz \) and \( yQw \). By Lemma 3.7, \( (x, z) \in Q \). This result allows to prove that if \( Q \) is reflexive then, \( \Phi' \) and \( \Phi \) are equivalent. For this is enough to show that \( \forall v (wQv \Rightarrow yQv) \Leftrightarrow yQw \). From left to right we use \( wQv \). For the other direction, suppose that \( yQw \) and \( wQv \), for every \( v \in X \) and use that \( \Phi' \) implies the transitivity of \( Q \).

Theorem 4.4. Let \( A \in \text{Hil}_\Box \) and let \( \langle X, K, Q \rangle \) be its dual space. Then:

1. \( A \vdash a \rightarrow \Box a \approx 1 \) iff \( \forall x \forall y (xQy \Rightarrow x \subseteq y) \).
2. \( A \vdash a \rightarrow \Box^n a \approx 1 \) iff \( \forall x \forall y (xQ^n y \Rightarrow x \subseteq y) \), with \( n \in \mathbb{N} \).
3. \( A \vdash \Box a \rightarrow a \approx 1 \) iff \( Q \) is reflexive.
4. \( A \vdash \Box a \rightarrow \Box^2 a \approx 1 \) iff \( Q \) is transitive.
5. \( A \vdash \Box^2 a \rightarrow \Box a \approx 1 \) iff \( Q \) is weakly dense, i.e., \( \forall x \forall y (xQy \Rightarrow \exists z (xQz \land zQy)) \).
6. \( A \vdash \Box (\Box a \rightarrow a) \approx 1 \) iff \( \forall x \forall y (xQy \Rightarrow yQy) \).
7. \( A \vdash (\Box a \rightarrow \Box b) \rightarrow \Box (\Box a \rightarrow \Box b) \approx 1 \) iff \( \langle X, K, Q \rangle \) satisfies \( \Phi \).
Proof. We will prove only the assertions (2), (5) and (7). The other proofs are analogous.

(2) Let \( n \in \mathbb{N} \). Suppose that there exist \( x, y \in X \) such that \((x, y) \in Q^n\) and \( x \nless y \). Hence, there is an element \( a \in x \) such that \( a \nless y \). As \((x, y) \in Q^n\) and \( a \nless y \), by Lemma 4.2, \( \square^n a \nless x \). Since \( a \leq \square^n a \), we have that \( a \nless x \), which is a contradiction. Reciprocally, if there exists \( a \in A \) such that \( a \nless \square^n a \) then, there exists \( x \in X \) such that \( a \in x \) and \( \square^n a \nless x \). By Lemma 4.2, we get an irreducible implicational filter \( y \in X \) such that \((x, y) \in Q^n\) and \( a \nless y \). By assumption, \( x \subseteq y \) and so, \( a \nless x \), which is impossible.

(5) Assume that \( \square^2 a \leq \square a \) for all \( a \in A \) and let \((x, y) \in Q \). Consider the implicational filter \( \square^{-1}(x) \) and the order-ideal \( (\square(y^c)) \). Suppose that there exists \( a \in \square^{-1}(x) \cap (\square(y^c)) \). So, \( \square a \in x \) and there exists \( p \in y^c \) such that \( a \leq \square p \). Thus, \( \square a \leq \square^2 p \leq \square p \) and consequently, \( \square p \in x \). So, \( p \in \square^{-1}(x) \). As \((x, y) \in Q\), we have that \( p \in y \), which is impossible. So, \( \square^{-1}(x) \cap (\square(y^c)) = \emptyset \). Thus, by Theorem 2.2, there exists \( z \in X \) such that \( \square^{-1}(x) \subseteq z \) and \( z \cap (\square(y^c)) = \emptyset \). This is, \( z \subseteq \square(y^c) \) and so, \( \square^{-1}(z) \subseteq y \).

Reciprocally, suppose that there exists \( a \in A \) such that \( \square^2 a \nless \square a \). So, there exists \( x \in X \) such that \( \square^2 a \in x \) and \( \square a \nless x \). By Lemma 4.2, there exists \( y \in X \) such that \((x, y) \in Q \) and \( a \nless y \). By assumption, \((x, y) \in Q^2\) and as \( a \nless y \), we get that \( \square^2 a \nless x \), which is a contradiction.

(7) Consider that \( (\square a \rightarrow \square b) \leq (\square a \rightarrow \square b) \), for every \( a, b \in A \). Let \((x, y) \in Q\) and \((y, z) \in Q\). Note that the implicational filter \( \langle x \cup \square(\square^{-1}(y)) \rangle \)
and the order-ideal \( (\square(z^c)) \) are disjoint. Indeed, suppose that there exists \( a \in A \) such that \( a \in \langle x \cup \square(\square^{-1}(y)) \rangle \) and \( a \in (\square(z^c)) \). Thus, by the characterization of implicational filter generated by a set given on page 50, there exist \( b \in x \), \( c \in \square^{-1}(y) \), and \( d \nless z \) such that \( b \rightarrow (\square c \rightarrow a) = 1 \) and \( a \leq d \). So, we have that \( 1 = b \rightarrow (\square c \rightarrow a) \leq b \rightarrow (\square c \rightarrow d) \). Then, \( b \rightarrow (\square c \rightarrow \square d) = 1 \in x \). Thus, \( \square c \rightarrow \square d \in x \). As \( \square c \rightarrow \square d \leq \square(\square c \rightarrow \square d) \), we get that \( \square(\square c \rightarrow \square d) \in x \). So, \( \square c \rightarrow \square b \in \square^{-1}(x) \) and by assumption, \( \square c \rightarrow \square d \in y \). As \( \square c \in y \), we get that \( \square d \in y \) and so, \( d \in z \), which is a contradiction. Thus, by Theorem 2.2 we can affirm that there exists \( w \in X \) such that \( x \subseteq w \), \( \square(\square^{-1}(y)) \subseteq w \) and \( \square(z^c) \cap w = \emptyset \). Hence, \( \square^{-1}(y) \subseteq \square^{-1}(w) \) and \( \square^{-1}(w) \subseteq z \). For every \( v \in X \) such that \((w, v) \in Q\), we get that \( \square^{-1}(y) \subseteq \square^{-1}(w) \subseteq v \). So, \((y, v) \in Q\). We have proved that \( \langle X, \mathcal{K}, Q \rangle \) satisfies the condition \( \Phi \).

Conversely, suppose that there exist \( a, b \in A \) such that \( \square a \rightarrow \square b \nless \)
\(\Box(\Box a \rightarrow \Box b)\). So, there exists \(x \in X\) such that \(\Box a \rightarrow \Box b \in x\) and 
\(\Box(\Box a \rightarrow \Box b) \notin x\). Then, there exists \(y \in X\) such that \(\Box^{-1}(x) \subseteq y\) and 
\(\Box a \rightarrow \Box b \notin y\). By consequence of Theorem 2.2, there exists \(z \in X\) such 
that \(y \subseteq z\), \(\Box a \in z\) and \(\Box b \notin z\). So, there exists \(w \in X\) such that 
\(\Box^{-1}(z) \subseteq w\) and \(b \notin w\). Thus, \((x, z) \in Q\) and \((z, w) \in Q\). By assumption, 
there exists \(v \in X\) such that \(x \subseteq v\), \((v, w) \in Q\) and for all \(u \in X\) such that 
\((v, u) \in Q\), we can affirm that \((z, u) \in Q\). Since \(\Box a \rightarrow \Box b \in x\), we have that 
\(\Box a \rightarrow \Box b \notin v\). On the other hand, \(b \notin w\) and so, \(\Box b \notin v\). Thus, \(\Box a \notin v\) and 
consequently, there exists \(u \in X\) such that \((v, u) \in Q\) and \(a \notin u\). Hence, 
\((z, u) \in Q\), and so, \(\Box a \notin z\), which is impossible. \(\Box\)

We shall say that an \(H\Box\)-algebra \(A, \Box\) is \textit{bounded} if the Hilbert algebra \(A\) is bounded. The variety of bounded \(H\Box\)-algebras is denoted by \(\text{Hil}^0\Box\).

**Theorem 4.5.** Let \(A \in \text{Hil}^0\Box\) and let \(\langle X, \mathcal{K}, Q \rangle\) be its dual space. Then,

1. \(A \models \Box 0 \rightarrow 0 \approx 1\) iff \(Q\) is serial, i.e., \(\forall x \exists y (xQy)\).

2. If \(Q\) is reflexive and transitive, we have that \(A \models \neg \Box a \rightarrow \Box \neg \Box a \approx 1\) 
iff \(Q \subseteq \emptyset \circ Q^{-1}\).

**Proof.** (1) Suppose that \(\Box 0 = 0\). Since \(0 \notin x\) for all \(x \in X\), we get that 
\(0 \notin \Box^{-1}(x)\). Thus, for each \(x \in X\) there exists \(y \in X\) such that \(\Box^{-1}(x) \subseteq y\) 
and \(0 \notin y\). So, \(Q\) is serial. Conversely, suppose that \(\Box 0 \notin 0\). There is 
\(x \in X\) such that \(\Box 0 \in x\) and \(0 \notin x\). Hence, \(0 \in \Box^{-1}(x)\) and by assumption, 
there exists \(y \in X\) such that \(\Box^{-1}(x) \subseteq y\). Thus, \(0 \in y\), which is impossible.

(2) Let \(Q\) be reflexive and transitive. Assume that \(\neg \Box a \leq \Box \neg \Box a\) for 
all \(a \in A\) and let \((x, y) \in Q\). Suppose that \(0 \in \langle x \cup \Box^{-1}(y) \rangle\). So, 
there exist \(a \in x\) and 
\(b \in \Box^{-1}(y)\) such that \(a \rightarrow (\Box b \rightarrow 0) = 1\), this is, 
a \leq \Box b\). Thus, \(\neg \Box b \in x\) and so, \(\Box \neg \Box b \in x\). Thus, \(\neg \Box b \in \Box^{-1}(x)\) and 
consequently, \(\Box b \rightarrow 0 \in y\). As \(\Box b \in y\), then \(0 \in y\), which is impossible. So, 
there exists \(z \in X\) such that \(\langle x \cup \Box^{-1}(y) \rangle \subseteq z\) and \(0 \notin z\). Hence, \(x \subseteq z\) 
and \(\Box^{-1}(y) \subseteq z\). So, \(\Box^{-1}(y) \subseteq \Box^{-1}(z)\). As \(Q\) is reflexive, \(\Box^{-1}(z) \subseteq z\) 
and so, \((y, z) \in Q\). Thus, \((x, y) \in \emptyset \circ Q^{-1}\).

Reciprocally. Assume that there is an element \(a \in A\) such that \(\neg \Box a \notin \Box \neg \Box a\). So, there exist \(x, y \in X\) such that \(\neg \Box a \in x\), \(\Box \neg \Box a \notin x\), \(\Box^{-1}(x) \subseteq y\) 
and \(\neg \Box a \notin y\). By Lemma 2.3, we have an irreducible implicative filter \(z\) 
such that \(y \subseteq z\) and \(\Box a \in z\). Thus, \((x, z) \in Q\) and \(\Box a \in z\). By assumption, 
there exists \(w \in X\) such that \(x \subseteq w\) and \((z, w) \in Q\). As \(\neg \Box a \in x\), we
have \(\neg \Box a \in w\). So, \(\Box a \notin w\), implying that \(\Box^2 a \notin z\). As \(Q\) is transitive, by Theorem 4.4, we have that \(\Box a \leq \Box^2 a\). So, \(\Box a \notin z\), which is impossible. 

We shall identify some subvarieties of \(\text{Hil}^0\): 

\[
\begin{align*}
\text{Hil}^0\text{S5} & = \text{Hil}^0 + \{T, 5\}, \\
\text{Hil}^0\text{S5.1} & = \text{Hil}^0 + \{T, 4, \neg \Box a \rightarrow \Box \neg \Box a \approx 1\}, \\
\text{Hil}^w\text{S5} & = \text{Hil}^0 + \{5, \Box 0 \rightarrow 0 \approx 1\}.
\end{align*}
\]

Note that \(\text{Hil}^0\text{S5}\) is subvariety of \(\text{Hil}^0\text{S5.1}\) and \(\text{Hil}^w\text{S5}\). Indeed. If \(A \in \text{Hil}^0\text{S5}\), we have that \(\Box a \rightarrow a \approx 1\), in particular, \(\Box 0 \rightarrow 0 \approx 1\). Thus, \(A \in \text{Hil}^0\text{S5}\). Moreover, by Remark 4.1, \(\Box a \rightarrow \Box^2 a \approx 1\) and as for all \(a \in A\), 

\[1 \equiv (\Box a \rightarrow 0) \rightarrow \Box (\Box a \rightarrow 0) = \neg \Box a \rightarrow \Box \neg \Box a\ ,\]

we get that \(A \in \text{Hil}^0\text{S5.1}\).

It is clear that \(\text{Hil}^0\text{S5.1}\) is subvariety of \(\text{Hil}^0\text{S4}\) and consequently, \(\text{Hil}^0\text{S5}\) is subvariety of \(\text{Hil}^0\text{S4}\).

**Corollary 4.6.** Let \(A \in \text{Hil}^0\) and \((X, K, Q)\) be its dual space. Then, 

\[A \in \text{Hil}^0\text{S5.1}\] 

iff \(Q\) is reflexive, transitive and \(Q \subseteq (\subseteq \circ Q^{-1})\).

**Proof.** By Theorem 4.4 and previous Theorem.  

5. Implicational modal logics

In this section we shall define the \(\{\rightarrow, \Box\}\)-fragment of the intuitionistic normal modal logic \(\text{IntK}^\Box\) and some of its extensions. Let \(\mathcal{L}\) be the propositional modal language with an infinite set of propositional variables \(Var\), a propositional constant \(\top\), the connective \(\rightarrow\), and the unary operator \(\Box\). The set of all formulas of \(\mathcal{L}\), we denote by \(Fm\).

The logic \(\text{IntK}^{\square}\) is a logic in the language \(\mathcal{L}^\Box\) characterized by the following list of axioms and rules:

1. \(\phi \rightarrow (\psi \rightarrow \phi)\),
2. \((\phi \rightarrow (\psi \rightarrow \alpha)) \rightarrow ((\phi \rightarrow \psi) \rightarrow ((\phi \rightarrow \alpha))\),
3. \(\Box(\phi \rightarrow \psi) \rightarrow (\Box \phi \rightarrow \Box \psi)\),
\[ \frac{\phi, \phi \rightarrow \psi}{\psi} \quad (\text{N}) \quad \frac{\phi \rightarrow \psi}{\Box \phi \rightarrow \Box \psi}. \]

It is clear that \( \text{IntK}^+ \) is the \( \{\Box, \rightarrow\} \)-fragment of intuitionistic modal logic \( \text{IntK} \). An implicational modal logic \( \mathcal{I} \) is any extension of \( \text{IntK}^+ \). Let \( \mathcal{F} = \langle X, \mathcal{K}, Q \rangle \) be an \( H\Box \)-frame or a general \( H\Box \)-frame (see Definition 3.3). A valuation on \( \mathcal{F} \) is a function \( V : \text{Var} \rightarrow \text{Up}(X) \) \( (V : \text{Var} \rightarrow D(X)) \) on the \( H\Box \)-frame (general \( H\Box \)-frame) \( \mathcal{F} \). As is usual, \( V \) is extended recursively to algebra of all formulas \( Fm \) by means of the clauses

1. \( V(\top) = X \),
2. \( V(\phi \rightarrow \psi) = V(\phi) \Rightarrow_{\mathcal{K}} V(\psi) = \text{sat}(V(\phi) \cap V(\psi)^c)^c \), and
3. \( V(\Box \phi) = \Box Q(\phi) = \{ x \in X : Q(x) \subseteq V(\phi) \} \).

By a general model we shall mean a structure \( \langle X, \mathcal{K}, Q, V \rangle \) where \( F = \langle X, \mathcal{K}, Q \rangle \) is an \( H\Box \)-frame or a general \( H\Box \)-frame and \( V \) is a valuation on \( \mathcal{F} \). We note that a function \( V \) is a valuation in an \( H\Box \)-frame or a general \( H\Box \)-frame \( \mathcal{F} \) iff it is a homomorphism between the algebra of all formulas \( Fm \) and \( A(\mathcal{F}) \) \( (D(X)) \). Then we get that a formula \( \phi \) is valid in an \( H\Box \)-frame (general \( H\Box \)-frame) \( \mathcal{F} \) iff the equation \( \phi \approx 1 \) is valid in the Hilbert algebra \( A(\mathcal{F}) \) \( (D(X)) \). Thus, we have that if \( \mathcal{F} \) is an \( H\Box \)-frame (general \( H\Box \)-frame),

\[ \mathcal{F} \models \phi \text{ iff } A(\mathcal{F}) \models \phi \approx 1 \text{ (} D(X) \models \phi \approx 1 \). \]

Let \( \mathcal{I} \) be an implicational modal logic. Denote by \( \text{Fr}(\mathcal{I}) \) the class of all general \( H\Box \)-frames where the formulas of \( \mathcal{I} \) are valid. Let \( \text{HSp}(\mathcal{I}) \) be the class of all \( H\Box \)-spaces \( \mathcal{F} = \langle X, \mathcal{K}, Q \rangle \) such that \( \mathcal{F} \models \phi \), for all \( \phi \in \mathcal{I} \). Clearly the class \( \text{HSp}(\mathcal{I}) \) is a subclass of \( \text{Fr}(\mathcal{I}) \).

We shall say that implicational modal logic \( \mathcal{I} \) is characterized by a class \( \mathcal{F} \) of general \( H\Box \)-frames, when \( \phi \in \mathcal{I} \) iff \( \phi \) is valid in every general \( H\Box \)-frame \( \langle X, \mathcal{K}, Q \rangle \in \mathcal{F} \). Moreover, it is frame complete when \( \phi \in \mathcal{I} \) iff \( \phi \) is valid in every general \( H\Box \)-frame \( \mathcal{F} = \langle X, \mathcal{K}, Q \rangle \), for any \( \mathcal{F} \in \text{Fr}(\mathcal{I}) \). It is clear that an implicational modal logic \( \mathcal{I} \) is frame complete if and only if it is characterized by some class of general \( H\Box \)-frames.

Let \( \mathcal{I} \) be an implicational modal logic. Consider the variety of Hilbert modal algebras \( V(\mathcal{I}) = \{ A \in \text{Hil} : A \models \phi \approx 1, \text{ for all } \phi \in \mathcal{I} \} \). Simple arguments (as in classical modal logic) show that

\[ \mathcal{F} \in \text{HSp}(\mathcal{I}) \text{ iff } D(X) \in V(\mathcal{I}). \]
Thus, we have the following result.

**Proposition 5.1.** Every implicational modal logic \( \mathcal{I} \square \) is characterized by the class \( \text{HSp}(\mathcal{I} \square) \).

Let \( \mathcal{F} = (X, K, Q) \) be a general \( H \square \)-frame. As \( D(X) \) is a subalgebra of \( A(\mathcal{F}) \), every formula valid in \( A(\mathcal{F}) \) is valid in \( D(X) \), but the converse in general is not valid.

**Definition 5.2.** We say that the variety \( \mathcal{V} \) of \( H \square \)-algebras is canonical, if \( A(\mathcal{F}(A)) \in \mathcal{V} \), when \( A \in \mathcal{V} \). An implicational modal logic \( \mathcal{I} \square \) is canonical if the variety \( \mathcal{V}(\mathcal{I} \square) \) is canonical.

An implicational modal logic \( \mathcal{I} \square \) is \( H \)-persistent if \( A(\mathcal{F}) \in \mathcal{V}(\mathcal{I} \square) \), for every \( H \square \)-space \( F = (X, K, Q) \).

The notion of implicational \( H \)-persistent modal logic is a generalization of the notion of \( d \)-persistent modal logic of classical modal logic (see [3] and [25]). By the results on duality between \( H \square \)-spaces and modal Hilbert algebras, we can give the following result.

**Proposition 5.3.** An implicational modal logic \( \mathcal{I} \square \) is \( H \)-persistent if and only if it is canonical.

**Proof.** Suppose that \( \mathcal{I} \square \) is \( H \)-persistent. Let \( A \in \mathcal{V}(\mathcal{I} \square) \). As \( A \) is isomorphic to \( D(X(A)) \), we have \( D(X(A)) \in \mathcal{V}(\mathcal{I} \square) \). As \( \mathcal{I} \square \) is \( H \)-persistent and taking into account that \( A(\mathcal{F}(D(X(A)))) \) is isomorphic to \( A(\mathcal{F}(A)) \), we get that \( A(\mathcal{F}(A)) \in \mathcal{V}(\mathcal{I} \square) \). So, \( \mathcal{I} \square \) is canonical.

For the converse we take an \( H \square \)-space \( \mathcal{F} = (X, K, Q) \), and suppose that \( D(X) \in \mathcal{V}(\mathcal{I} \square) \). As \( \mathcal{F} \) is an \( H \square \)-space, \( X \) is homeomorphic (and also order-isomorphic) to \( X(D(X)) \). Then \( \text{Up}(X) \) is isomorphic to \( \text{Up}(X(D(X))) \). Thus the Hilbert modal algebras \( A(\mathcal{F}) \) and \( A(\mathcal{F}(D(X))) \) are isomorphic, and consequently \( A(\mathcal{F}) \in \mathcal{V}(\mathcal{I} \square) \). \( \square \)

**Proposition 5.4.** Every canonical implicational modal logic \( \mathcal{I} \square \) is complete with respect to \( \text{Fr}(\mathcal{I} \square) \).

**Proof.** The proof is as in classical modal logic. We need to prove that for each formula \( \phi \not\in \mathcal{I} \square \) there exists an \( H \square \)-frame \( \mathcal{F} \) of \( \mathcal{I} \square \) such that \( \phi \) is refuted in \( \mathcal{F} \). Let \( \phi \not\in \mathcal{I} \square \). Then there exists a modal Hilbert algebra \( A \) such that \( A \not\models \phi \approx 1 \). Then there exists a homomorphism \( h : \text{Fm} \rightarrow A \)
such that $h(\phi) \neq 1$. By Theorem 2.2 there exists $x \in X(A)$ such that
$h(\phi) \notin x$. Let $F(A) = \langle X(A), K_A, Q_A \rangle$ be the $H\square$-frame of $A$. As $\mathcal{I}_\square$ is
canonical, $A(F(A)) \in \mathcal{V}(\mathcal{I}_\square)$, i.e., $F(A)$ is an $H\square$-frame of $\mathcal{I}_\square$. As the map
$\varphi : A \to D(X(A))$ is an one to one homomorphism, the composition $\varphi \circ h$ is
a homomorphism from $Fm$ into $D(X(A))$, i.e., $\varphi \circ h$ is a valuation based on $F(A)$. So, $(\varphi \circ h)(\phi) = \varphi(h(\phi)) \neq \varphi(1) = X(A)$, because $x \notin \varphi(h(\phi))$. So the formula $\phi$ is refuted in the general model
$(X(A), K_A, \varphi \circ h)$. Therefore, $\phi$ is refuted in the $H\square$-frame $F(A)$.

Given the characterizations proved in the Section 4, we can ensure that
any variety of $H\square$-algebras axiomatized by some subset of the set of equations:

$$ \{S, S_n, T, wD, 4, 5, 6, \square 0 \to 0 \approx 1, -\square a \to \square -\square a \approx 1, \square(\square a \to a) \approx 1\} $$

is canonical. Therefore we obtain the following result.

**Theorem 5.5.** Any variety of $H\square$-algebras axiomatized by formulas
belong to $P$ are canonical. Therefore, the associated logics are canonical
and frame complete.

6. Simple and subdirectly irreducibles $H\square$-algebras

Denote by $\text{Con}(A, \to)$ the lattice of all congruences on a Hilbert algebra $A$
and call the set $[1]_\theta = \{x \in A : (x, 1) \in \theta\}$ the kernel of $\theta$. If $D \in \text{Fi}(A)$
then the binary relation $\theta_D$ defined by

$$(a, b) \in \theta_D \text{ iff } a \to b \in D \text{ and } b \to a \in D$$

is a congruence on $A$ such that $[1]_\theta = D$. Moreover, the lattices $\text{Fi}(A)$ and
$\text{Con}(A, \to)$ are isomorphic under the mutually inverse mappings $\theta \to [1]_\theta$
and $D \to \theta_D$ (see [11], [15], or [18]).

Let $A \in \text{Hil}_{\square}$. Denote by $\text{Con}(A, \to, \square)$ the lattice of congruences of $A$. Let $F \in \text{Fi}(A)$. We said that $F$ is a $\square$-implicative filter if $\square a \in F$,
whenever $a \in F$, i.e., $F \subseteq \square^{-1}(F)$. The set of all $\square$-implicative filters of
an $H\square$-algebra $A$ is denoted by $\text{Fi}_{\square}(A)$.

Let $n \in \mathbb{N}_0$. We define the symbol

$$(\alpha_n(a); b) = (a, \square a, ..., \square^n a; b)$$
for all \( a, b \in A \). For each non-empty subset \( X \) of \( A \), we define the set \( \langle X \rangle \Box \) as:

\[
\langle X \rangle \Box = \{ a \in A : \exists x_1, \ldots, x_k \in X, n_1, \ldots, n_k \in \mathbb{N}_0 \quad [(\alpha_{n_1}(x_1); \ldots; (\alpha_{n_k}(x_k); a); \ldots) = 1] \}.
\]

Note that if \( X = \{ a \} \), then

\[
\langle \{ a \} \rangle \Box = \langle a \rangle \Box = \{ b \in A : \exists n \in \mathbb{N}_0 : (\alpha_n(a); b) = 1 \}.
\]

**Remark 6.1.** As any Hilbert algebra \( A \) satisfies the Change Law, i.e.,

\[
a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c)
\]

for all \( a, b, c \in A \), we get that any \( H \Box \)-algebra \( \langle A, \Box \rangle \) satisfies the identity

\[
(\alpha_{n_1}(a); (\alpha_{n_2}(b); c)) = (\alpha_{n_2}(b); (\alpha_{n_1}(a); c))
\]

for all \( a, b, c \in A, n_1, n_2 \in \mathbb{N}_0 \).

Moreover, note that if \( A \in \text{Hil}_{\Box} \) and \( a, b \in A \) such that \( a \leq b \), then

\[
(\alpha_n(x); a) \leq (\alpha_n(x); b),
\]

for all \( x, a \in A, n \in \mathbb{N}_0 \).

**Lemma 6.2.** Let \( A \in \text{Hil}_{\Box} \). Then,

\[
x \rightarrow \Box(\alpha_n(x); a) \leq (\alpha_{n+1}(x); \Box a),
\]

for all \( x, a \in A, n \in \mathbb{N}_0 \).

**Proof.** By Definition 3.1,

\[
\Box(\alpha_n(x); a) = \Box(x, \Box x, \ldots, \Box^n x; a)
\]

\[
\leq \Box x \rightarrow \Box(\Box x, \ldots, \Box^n x; a)
\]

\[
\leq \Box x \rightarrow (\Box^2 x \rightarrow (\Box^3 x \rightarrow \ldots(\Box^{n+1} x \rightarrow \Box a)\ldots)).
\]

Thus,

\[
x \rightarrow \Box(\alpha_n(x); a) \leq x \rightarrow (\Box x \rightarrow (\Box^2 x \rightarrow \ldots(\Box^{n+1} x \rightarrow \Box a)\ldots))
\]

\[
= (\alpha_{n+1}(x); \Box a).
\]

\[\square\]

**Corollary 6.3.** Let \( A \in \text{Hil}_{\Box} \). Then,

\[
x_k \rightarrow (x_{k-1} \rightarrow \ldots (x_1 \rightarrow \Box [(\alpha_{n_1}(x_1); (\ldots (\alpha_{n_k}(x_k); a); \ldots)] \ldots) \leq
\]

\[
(\alpha_{n_1+1}(x_1); (\ldots (\alpha_{n_k+1}(x_k); \Box a); \ldots))
\]

for all \( k \in \mathbb{N}, a, x_1, \ldots, x_k \in A, n_1, \ldots, n_k \in \mathbb{N}_0 \).
**Proof.** By Lemma 6.2,

\[ x_k \rightarrow □(α_{n_k}(x_k); a) \leq (α_{n_k+1}(x_k); □a). \]

So, by above Remark,

\[ (α_{n_k-1}(x_k); (x_k \rightarrow □(α_k(x_k); a))) \leq (α_{n_k-1+1}(x_k-1); (α_{n_k+1}(x_k); □a)) \]

and by Chance Law,

\[ x_k \rightarrow (α_{n_k-1+1}(x_k-1); □(α_k(x_k); a)) \leq (α_{n_k-1+1}(x_k-1); (α_{n_k+1}(x_k); □a)). \]

By Lemma 6.2,

\[ x_{k-1} \rightarrow □(α_{n_k-1+1}(x_k-1); (α_k(x_k); a)) \leq (α_{n_k-1+1}(x_k-1); □(α_k(x_k); a)). \]

So,

\[ x_k \rightarrow (x_{k-1} \rightarrow □(α_{n_k-1+1}(x_k-1); (α_k(x_k); a)) \leq x_k \rightarrow (α_{n_k-1+1}(x_k-1); □(α_k(x_k); a)) \leq (α_{n_k-1+1}(x_k-1); (α_{n_k+1}(x_k); □a)). \]

Repeating this procedure we obtain that

\[ x_k \rightarrow (x_{k-1} \rightarrow \ldots (x_1 \rightarrow □[(α_{n_1}(x_1); (\ldots (α_{n_k}(x_k); a))\ldots)]) \ldots) \leq (α_{n_1+1}(x_1); (\ldots (α_{n_k+1}(x_k); □a))\ldots). \]

□

**Lemma 6.4.** Let \( A \in \text{Hil}_□ \) and \( X \subseteq A \). Then, \( \langle X \rangle_□ \) is the smallest \( □ \)-implicative filter containing to \( X \).

**Proof.** It is clear that \( \langle X \rangle_□ \in \text{Fi}(A) \). Let \( a \in \langle X \rangle_□ \). So, there exists \( k \in \mathbb{N} \) and there exist \( x_1, \ldots, x_k \in X, n_1, \ldots, n_k \in \mathbb{N}_0 \) such that

\[ (α_{n_1}(x_1); (α_{n_2}(x_2); (\ldots (α_{n_k}(x_k); a))\ldots)) = 1. \]

Hence, \( □(α_{n_1}(x_1); (α_{n_2}(x_2); (\ldots (α_{n_k}(x_k); a))\ldots)) = □1 = 1 \). So,

\[ x_k \rightarrow (x_{k-1} \rightarrow \ldots (x_1 \rightarrow □(α_{n_1}(x_1); (\ldots (α_{n_k}(x_k); a))\ldots)) \ldots) = 1. \]
Thus, by above Corollary, \(1 \leq (\alpha_{n_1+1}(x_1); (\ldots (\alpha_{n_k+1}(x_k); \Box a)) \ldots)\) and consequently,
\[
(\alpha_{n_1+1}(x_1); (\ldots (\alpha_{n_k+1}(x_k); \Box a)) \ldots) = 1,
\]
with \(x_1, \ldots, x_k \in X\) and \(n_1 + 1, \ldots, n_k + 1 \in \mathbb{N}_0\). Consequently, \(\Box a \in \langle X \rangle\) and so, \(\langle X \rangle \subseteq \text{Fi}\Box(A)\).

Finally, it is easy to see that if \(F \in \text{Fi}\Box(A)\) and \(X \subseteq F\), then \(\langle X \rangle \subseteq F\).

\[\square\]

In some subvarieties of \(\text{Hil}\Box\) we can give simplified expressions of \(\langle X \rangle\).

If \(A \in \text{Hil}\Box + \{4\}\), then
\[
(\alpha_n(a); b) = (\alpha_1(a); b) \quad (4)
\]
for all \(a, b \in A\), and for all \(n \in \mathbb{N}\). If \(A \in \text{Hil}\Box \text{S4}\), then,
\[
(\alpha_n(a); b) = \Box a \rightarrow b, \quad (5)
\]
for all \(a, b \in A\), and for all \(n \in \mathbb{N}\).

\textbf{Definition 6.5.} Let \(\langle X, \mathcal{K}, Q \rangle\) be an \(H\Box\)-space. A subset closed \(Y\) of \(X\) will be called \(Q\)-closed if \(Q(Y) = \bigcup \{Q(y) : y \in Y\} \subseteq Y\).

The set of all \(Q\)-closed subsets of an \(H\Box\)-space \(\langle X, \mathcal{K}, Q \rangle\) is denoted by \(\mathcal{C}_Q(X)\).

If \(L\) is a lattice, \(L^d\) is the lattice with the dual order. Let \(L_1\) and \(L_2\) be two lattices. If two lattices \(L_1\) and \(L_2\) are isomorphic we write \(L_1 \cong L_2\).

\textbf{Proposition 6.6.} Let \(A \in \text{Hil}\Box\) and let \(\langle X, \mathcal{K}, Q \rangle\) be its dual space. Then,
\[
\text{Con} (A, \rightarrow, \Box) \cong \text{Fi}\Box(A) \cong \mathcal{C}_Q(X)^d.
\]

\textbf{Proof.} Let \(\theta \in \text{Con} (A, \rightarrow, \Box)\). It is clear that \([1]_\theta \in \text{Fi}\Box(A)\). Now, let \(F \in \text{Fi}\Box(A)\). We know that \(\theta F \in \text{Con} (A, \rightarrow)\). If \((a, b) \in \theta F\) then \(a \rightarrow b, b \rightarrow a \in F\). So, \(\Box (a \rightarrow b), \Box (b \rightarrow a) \in F\). As \(\Box (a \rightarrow b) \leq \Box a \rightarrow \Box b\), we get that \(\Box a \rightarrow \Box b \in F\). Analogously, \(\Box b \rightarrow \Box a \in F\) and so, \((\Box a, \Box b) \in \theta F\).

We will prove that \(\text{Fi}\Box(A) \cong \mathcal{C}_Q(X)^d\). Let \(F \in \text{Fi}\Box(A)\). So,
\[
\delta(F) = \{x \in X : F \subseteq x \} = \bigcap \{\varphi(a) : a \in F\},
\]
is a closed subset of \(X\). Let \(y \in Q(\delta(F))\). So, exists \(x \in \delta(F)\) such that \(y \in Q(x)\). As \(F\) is a \(\Box\)-implicative filter, \(F \subseteq \Box^{-1}(F) \subseteq \Box^{-1}(x) \subseteq y\), and
hence, \( y \in \delta(F) \). Then \( \delta(F) \) is a \( Q \)-closed. Note that if \( F, H \in \text{Fi}_H(A) \) such that \( F \subseteq H \) then \( \delta(H) \subseteq \delta(F) \).

Now, we will prove that \( \pi : C_Q(X) \to \text{Fi}_H(A) \) given by
\[
\pi(Y) = \{a \in A : Y \subseteq \varphi(a)\}
\]
is well-defined. It is clear that \( \pi(Y) \in \text{Fi}(A) \). We prove that \( \pi(Y) \) is a \( \Box \)-implicative filter. Let \( a \in A \) such that \( Y \subseteq \varphi(a) \). As \( Y \) is \( Q \)-closed, \( Q(Y) \subseteq Y \subseteq \varphi(a) \). Suppose that \( Y \notin \varphi(\Box a) \). So, there exists \( x \in Y \) such that \( x \notin \varphi(\Box a) \). Thus, \( \Box a \notin x \) and so, there exists \( y \in X \) such that \( y \in Q(x) \) and \( a \notin y \). As \( x \in Y \), we get \( y \in Q(Y) \). Thus, \( y \in Y \) and \( y \notin \varphi(a) \), which is a contradiction. So, \( \pi(Y) \in \text{Fi}_H(A) \).

Next, we will prove that \( \delta \) and \( \pi \) are inverses of each other. Let \( Y \in C_Q(X) \). So,
\[
\delta(\pi(Y)) = \bigcap \{\varphi(a) : a \in \pi(Y)\} = Y.
\]
Now, let \( F \in \text{Fi}_H(A) \). Suppose that there exists \( a \in \pi(\delta(F)) = \{b \in A : \delta(F) \subseteq \varphi(b)\} \) such that \( a \notin F \), this is, \( (a) \cap F = \emptyset \). By Theorem 2.2, there exists \( x \in X \) such that \( F \subseteq x \) and \( a \notin x \), which contradicts the assumed. So, \( \pi(\delta(F)) \subseteq F \). On the other hand, as \( \delta(F) = \bigcap \{\varphi(a) : a \in F\} \subseteq \varphi(b) \) for every \( b \in F \), we have that \( F \subseteq \pi(\delta(F)) \). Thus, we deduce that \( \delta \) is a lattice anti-isomorphism.

Let \( A \in \text{Hil}_H \). Let us recall that \( A \) is subdirectly irreducible if and only if there exists the smallest non trivial \( \Box \)-congruence relation \( \theta \) in \( A \). And \( A \) is simple if and only if \( A \) has only two \( \Box \)-congruence relations. By Proposition 6.6 we have that \( A \) is subdirectly irreducible if there exists the smallest non-trivial \( \Box \)-implicative filter in \( A \) iff in its dual \( H\Box \)-space \( \langle X, K, Q \rangle \) there exists the largest \( Q \)-closed subset distinct from \( X \). Moreover, \( A \) is simple iff \( \text{Fi}(A) = \{\{1\}, A\} \) iff \( C_Q(X) = \{\emptyset, X\} \). Now, we give a new characterization of simple and subdirectly irreducible algebras in the variety \( \text{Hil}_H \).

**Lemma 6.7.** Let \( \langle X, K, Q \rangle \) be an \( H\Box \)-space. Then, \( V_x = \text{cl}(Q^*(x)) \) is the smallest \( Q \)-closed set containing the element \( x \).
Proof. As $Q^*$ is reflexive and $Q^*(x) \subseteq \text{cl}(Q^*(x))$ for each $x \in X$, we get that $x \in \text{cl}(Q^*(x))$. In addition, as $\text{cl}(Q^*(x))$ is a closed subset of $X$, only remains to prove that $Q(\text{cl}(Q^*(x))) \subseteq \text{cl}(Q^*(x))$ for each $x \in X$. Let $y \in X$ such that $y \in Q(\text{cl}(Q^*(x)))$. So, there exists $z \in \text{cl}(Q^*(x))$ such that $(z, y) \in Q$. Suppose that $y \notin \text{cl}(Q^*(x))$, then there exists $a \in A$ such that $\text{cl}(Q^*(x)) \subseteq \varphi(a)$ and $y \notin \varphi(a)$. Since $Q^*(x) \subseteq \text{cl}(Q^*(x)) \subseteq \varphi(a)$, we get that $Q^n(x) \subseteq \varphi(a)$ for all $n \geq 0$. This is, $a \in w$ for all $w \in Q^n(x)$. By Lemma 4.2, $\Box^n a \in x$ for all $n \geq 0$. On the other hand, as $a \notin y$, we get that $\Box a \notin z$ and since $z \in \text{cl}(Q^*(x))$, result $\varphi(\Box a)^c \cap Q^*(x) \neq \emptyset$. So, there exists $v \in X$ such that $(x, v) \in Q^m$ for some $m \geq 0$ and $\Box a \notin v$. By Lemma 4.2, $\Box^m a \notin x$ for some $m \geq 0$, which is impossible. Thus, $\text{cl}(Q^*(x)) \subseteq C_Q(X)$. Let $V \in C_Q(X)$ such that $x \in V$. Then $Q^n(x) \subseteq V$, for all $n \geq 0$, because $V$ is a $Q$-closed. So, $Q^*(x) = \bigcup \{Q^n(x) : n \geq 0\} \subseteq V$. Thus, $\text{cl}(Q^*(x)) \subseteq \text{cl}(V) = V$. 

We note that $\text{cl}(Q^*(x)) = \bigcap \{V : V \in C_Q(X) \text{ and } x \in V\}$.

Let $(X, K, Q)$ be an $H\Box$-space. Let us define the following subsets of $X$:

$$I_X = \{x \in X : V_x = X\} \text{ and } H_X = X - I_X,$$

where $V_x = \text{cl}(Q^*(x))$.

Our first main result characterizes the simple algebras as the ones of which the dual space is generated from each point.

**Theorem 6.8.** Let $A \in \text{Hil}_{\Box}$ and let $(X, K, Q)$ be its dual space. Then, the following conditions are equivalent:

1. $A$ is simple,
2. $I_X = X$, i.e., $V_x = X$, for each $x \in X$,
3. $(a)_{\Box} = A$, for all $a \in A - \{1\}$.

**Proof.** (1) \(\Rightarrow\) (2) By Lemma 6.7.

(2) \(\Rightarrow\) (3) Suppose that there exists $a \in A - \{1\}$ such that $(a)_{\Box} \neq A$. So, there exists $b \in A$ such that $b \notin (a)_{\Box}$. This is, $(\alpha_n(a); b) \neq 1$ for all $n \geq 0$. So, there exists $x \in X$ such that $\Box^n a \in x$ for all $n \geq 0$ and $b \notin x$. As $\text{cl}(Q^*(x)) = X$, we get that $\varphi(a)^c \cap Q^*(x) \neq \emptyset$. So, there exists $z \in Q^*(x)$ such that $a \notin z$. Hence, there exists $m \geq 0$ such that $(x, z) \in Q^m$ and $a \notin z$. By Lemma 4.2, $\Box^m a \notin x$, which is impossible.
(3) $\Rightarrow$ (1) Let $F \in \text{Fi}_\square(A)$. Let $a \in F$ such that $a \neq 1$. Then $\langle a \rangle_\square = A \subseteq F$. Thus, $F = A$, and consequently $\text{Fi}_\square(A) = \{\{1\}, A\}$. Thus, $A$ is simple. 

We note that the previous Theorem affirms that $A$ is an $H_\square$-algebra simple if and only if $H_X = \emptyset$.

Our second main result gives a similar characterization of the subdirectly irreducible algebras.

**Theorem 6.9.** Let $A \in \text{Hil}_\square$ and let $(X, \mathcal{K}, Q)$ be its dual space. Then, the following conditions are equivalent:

1. $A$ is subdirectly irreducible.
2. $H_X = \{x \in X \mid V_x \neq X\} \in \mathcal{C}_Q(X) - \{X\}$,
3. There exists $a \in A - \{1\}$ such that for all $b \in A - \{1\}$ there exists $n \geq 0$ such that $(\alpha_n(b); a) = 1$.

**Proof.** (1) $\Rightarrow$ (2) By assumption, there exists the largest $V \in \mathcal{C}_Q(X) - \{X\}$. We will prove that $V = H_X$. It is clear that $H_X \subseteq V$. Let $x \in V$. As $V \in \mathcal{C}_Q(X)$, by Lemma 6.7, $V_x \subseteq V$. Since $V \neq X$, $V_x \neq X$ and so, $x \in H_X$.

(2) $\Rightarrow$ (3) Since $H_X \neq X$, there exists $x \in X$ such that $x \notin H_X$. As $H_X$ is closed, there exists $a \in A - \{1\}$ such that $H_X \subseteq \varphi(a)$ and $x \notin \varphi(a)$. We will prove that for all $b \in A - \{1\}$ there exists $n \geq 0$ such that $(\alpha_n(b); a) = 1$. On the contrary, suppose that there exists $b \in A - \{1\}$ such that $(\alpha_n(b); a) \neq 1$ for all $n \geq 0$. So, there exists $w \in X$ such that $\square^n b \in w$ for all $n \geq 0$ and $a \notin w$. As $w \notin \varphi(a)$, we get that $w \notin H_X$ and consequently, $\text{cl}(Q^*(w)) = X$. Thus, $Q^*(w) \cap \varphi(b)^c \neq \emptyset$ and so, there exists $z \in Q^*(w)$ and $b \not\in z$. So, there exists $m \geq 0$ such that $(w, z) \in Q^m$ and $b \not\in z$. By Lemma 4.2, $\square^m b \notin w$, which is impossible.

(3) $\Rightarrow$ (1) By assumption, $a \in \langle b \rangle_\square$ for all $b \in A - \{1\}$. As $\langle b \rangle_\square \in \text{Fi}_\square(A)$, we have that $\langle a \rangle_\square \subseteq \langle b \rangle_\square$ for all $b \in A - \{1\}$. As $a \neq 1$, we get that $\langle a \rangle_\square \neq \{1\}$. We will prove that $\langle a \rangle_\square$ is the smallest non-trivial $\square$-implicative filter. Let $F \in \text{Fi}_\square(A) - \{1\}$. So, there exists $b \neq 1$ such that $b \in F$. As $\langle b \rangle_\square$ is the smallest $\square$-implicative filter containing to $b$, we get that $\langle a \rangle_\square \subseteq \langle b \rangle_\square \subseteq F$. Thus, $A$ is subdirectly irreducible.

Now, we shall study the simple and subdirectly irreducible algebras in the varieties Hil$_\square$S4, Hil$_\square$S4, Hil$_\square$S5.1, and Hil$_\square$S5.
Remark 6.10. Let \( A \in \text{Hil}_{\square S4} \) and let \( \langle X, \mathcal{K}, Q \rangle \) be its dual space.

(1) By items 3 and 4 of Theorem 4.4, we get that \( Q \) is transitive and reflexive. Thus, \( Q^*(x) = Q(x) \), for each \( x \in X \), and as \( Q(x) \) is a closed subset of \( X \), we have that \( Q(x) = V_x \), for each \( x \in X \).

(2) If \( H_X \neq \emptyset \), then \( H_X = \bigcup \{ \varphi(\square a) : a \in A - \{1\} \} \). Indeed:

\[
x \in H_X \quad \text{iff} \quad Q(x) = V_x \neq X
\]
\[
\quad \text{iff} \quad \exists y \in X : y \notin Q(x)
\]
\[
\quad \text{iff} \quad \exists y \in X \exists a \in A : Q(x) \subseteq \varphi(a) \land y \notin \varphi(a)
\]
\[
\quad \text{iff} \quad \exists y \in X \exists a \in A : x \in \square Q(\varphi(a)) = \varphi(\square a) \land a \notin y
\]
\[
\quad \text{iff} \quad x \in \bigcup \{ \varphi(\square a) : a \in A - \{1\} \}.
\]

The following result is a simple consequence of Theorem 6.8, item (1) of Remark 6.10 and the formula (5).

Proposition 6.11. Let \( A \in \text{Hil}_{\square S4} \) and let \( \langle X, \mathcal{K}, Q \rangle \) be its dual space.
Then, the following conditions are equivalent:

1. \( A \) is simple.
2. \( Q(x) = X \), for each \( x \in X \).
3. \( \langle \square a \rangle = A \) for all \( a \in A - \{1\} \). This is, \( A \) is bounded.

Proposition 6.12. Let \( A \in \text{Hil}_{\square S4} \) and let \( \langle X, \mathcal{K}, Q \rangle \) be its dual space.
Then, the following conditions are equivalent:

1. \( A \) is subdirectly irreducible.
2. \( H_X \in D(X) - \{X\} \).
3. There exists \( a \in A - \{1\} \) such that \( \square b \leq a \), for all \( b \in A - \{1\} \).

Proof. (1) \( \Rightarrow \) (2) By Theorem 6.9, \( H_X \in C_Q(X) - \{X\} \). So, exists \( x \in X \) such that \( x \notin H_X \). Thus, there exists \( c \in A - \{1\} \) such that \( H_X \subseteq \varphi(c) \) and \( x \notin \varphi(c) \). As in the proof of Proposition 6.6, if \( H_X \in C_Q(X) \) and \( H_X \subseteq \varphi(c) \) then, \( H_X \subseteq \varphi(\square c) \). If \( H_X \neq \emptyset \), by Remark 6.10, \( H_X = \bigcup \{ \varphi(\square b) : b \in A - \{1\} \} \). As \( c \neq 1 \), \( \varphi(\square c) \subseteq H_X \). Thus, \( H_X = \varphi(\square c) \in D(X) - \{X\} \).
(2) \(\Rightarrow\) (3) Let \(H_X \in D(X) - \{X\}\). So, there exists \(a \in A - \{1\}\) such that \(H_X = \varphi(a)\). If \(H_X = \emptyset\), then \(Q(x) = X\) for all \(x \in X\) and by Proposition 6.11, \((\square b) = A\) for all \(b \in A - \{1\}\). Let \(a \in A - \{1\}\). Then \(a \in (\square b)\) for all \(b \in A - \{1\}\). So, \(\square b \leq a\), for all \(b \in A - \{1\}\). If \(H_X \neq \emptyset\), by Remark 6.10, \(H_X = \bigcup \{\varphi(\square b) : b \in A - \{1\}\} = \varphi(a)\). Therefore, \(\varphi(\square b) \subseteq \varphi(a)\) and consequently, \(\square b \leq a\) for all \(b \in A - \{1\}\), because \(\varphi\) is an isomorphism.

(3) \(\Rightarrow\) (1) It is an immediate consequence of the formula (5) and Theorem 6.9.

\[\square\]

Corollary 6.13. Let \(A \in \mathsf{Hil}^0_{\mathbb{S}4}\) and let \((X, K, Q)\) be its dual space. Then,

1. \(A\) is simple iff \(\square a = 0\), for all \(a \in A - \{1\}\).

2. \(A\) is subdirectly irreducible iff \(H_X \in D(X) - \{X\}\) iff there exists \(a \in A - \{1\}\) for all \(b \in A - \{1\}\) such that \(\square b \leq a\).

**Proof.** (1) As \(A\) is bounded, \(A = \{0\}\). Thus, by Proposition 6.11, \(A\) is simple iff \((\square a) = \{0\}\) for \(a \in A - \{1\}\) iff \(\square a = 0\) for \(a \in A - \{1\}\).

(2) By Proposition 6.12.

\[\square\]

Proposition 6.14. Let \(A \in \mathsf{Hil}^0_{\mathbb{S}5.1}\). Then,

1. \(A\) is simple iff \(\square a = 0\), for all \(a \in A - \{1\}\).

2. \(A\) is subdirectly irreducible not simple iff there exists \(a \in A - \{1\}\) such that \(\square b \leq a\) and \(\neg \square a = 0\), for all \(b \in A - \{1\}\).

**Proof.** (1) By Corollary 6.13, because \(\mathsf{Hil}^0_{\mathbb{S}5.1}\) is subvariety of \(\mathsf{Hil}^0_{\mathbb{S}4}\).

(2) Let \(A\) be subdirectly irreducible. So, there exists \(a \in A - \{1\}\) such that \(\square b \leq a\), for all \(b \in A - \{1\}\). It remains to prove that \(A\) is not simple iff \(\neg \square a = 0\). If \(A\) is not simple then exists \(b \neq 1\) such that \(\square b \neq 0\), i.e., \(\square b \neq 0\). This is, \(\neg \square b \neq 1\) and so, \(\square \neg \square b \leq a\). Thus, \(\neg \square b \leq a\) and hence, \(\neg \square a \leq \neg \square b\). As any Hilbert algebra \(A\) satisfies \((c \to d) \to ((d \to c) \to c) = (d \to c) \to (c \to d) \to d\), replacing \(c\) by 0 result \(\neg \square d = \neg d \to d\). Thus, \(\neg \square a \leq \neg \square b \to \square b \leq \neg \square b \to b\) and so, \(\neg \square a \to (\neg \square b \to b) = (\neg \square a \to \neg \square b) \to (\neg \square a \to b) = 1\). As \(b \neq 1\), we have \(\square b \leq a\) and so, \(\square b = \square^2 b \leq \square a\). Thus, \(\neg \square a \to \neg \square b = 1\) and consequently, \(\neg \square a \to b = 1\). As \(\neg \square a \leq b \neq 1\), we get that \(\neg \square a \neq 1\) and so, \(\neg \square a \leq \neg \square a \leq a\).
Hence, \((\alpha_0(\neg\Box a); a) = 1\) and thus, \(a \in \langle \neg\Box a \rangle\). As \(\langle \neg\Box a \rangle \in \text{Fi}_{\Box}(A)\), \(\Box a \in \langle \neg\Box a \rangle\) and so, \(0 \in \langle \neg\Box a \rangle\). Thus, \(\neg\Box a = 0\). Reciprocally, if there exists \(a \neq 1\) such that \(\neg a = 0\), then \(\Box a \not\rightarrow 0 \neq 1\). This is, \(\Box a \not\in 0\) and so, \(\Box a \neq 0\). Thus, \(A\) is not simple. \(\Box\)

**Lemma 6.15.** Let \(A \in \text{Hil}_{\Box}^{\text{S5}}\). Then, \(\langle a \rangle_{\Box} = \{ b : a \rightarrow (\Box a \rightarrow b) = 1 \}\).

**Proof.** It is easy and left to the reader. \(\Box\)

**Proposition 6.16.** Let \(A \in \text{Hil}_{\Box}^{\text{S5}}\). Then,

1. \(A\) is simple iff \(\Box a = 0\), for all \(a \in A - \{1\}\).
2. \(A\) is subdirectly irreducible iff there exists \(a \in A - \{1\}\) such that \((\alpha_1(b); a) = 1\) for all \(b \in A - \{1\}\).

**Proof.** Let \(A \in \text{Hil}_{\Box}^{\text{S5}}\). By Remark 4.1, \(\Box a \leq \Box^2 a\) for all \(a \in A\).

1. \((\Rightarrow)\) Let \(a \in A\). As \(\Box a \leq \Box^2 a\), we get that \(\Box b \in \langle \Box a \rangle\) when \(b \in \langle \Box a \rangle\). Thus, \(\langle \Box a \rangle \in \text{Fi}_{\Box}(A)\). As \(A\) is simple, \(\langle \Box a \rangle = A\) or \(\langle \Box a \rangle = \{1\}\). This is, \(\Box a = 0\) or \(\Box a = 1\). The proof is completed by showing that \(\Box a = 1\) iff \(a = 1\). Suppose that there exists \(a \neq 1\) such that \(\Box a = 1\). As \(A\) is simple, by Theorem 6.8, \(\langle a \rangle_{\Box} = A\). Note that \(\langle a \rangle_{\Box} = \langle a \rangle\). In fact, it is clear that \(\langle a \rangle \subseteq \langle a \rangle_{\Box}\). Let \(b \in \langle a \rangle_{\Box}\). By Lemma 6.15 we have \(1 = a \rightarrow (\Box a \rightarrow b) = a \rightarrow (1 \rightarrow b) = a \rightarrow b\). So, \(b \in \langle a \rangle\). Thus, \(A = \langle a \rangle_{\Box}\), and consequently \(a = 0\). Thus, \(\Box a = 0\) which is impossible.

\((\Leftarrow)\) It is clear that \(\Box a \in \langle a \rangle_{\Box}\). So, \(\langle \Box a \rangle \subseteq \langle a \rangle_{\Box}\) for all \(a \in A\). By assumption, \(A = \langle 0 \rangle = \langle \Box a \rangle \subseteq \langle a \rangle_{\Box}\) for \(a \in A - \{1\}\) and consequently \(A = \langle a \rangle_{\Box}\), for \(a \in A - \{1\}\). Then by Theorem 6.8, \(A\) is simple.

2. By Theorem 6.9, there exists \(a \in A - \{1\}\) such that for all \(b \in A - \{1\}\) there exists \(n \geq 0\) such that \((\alpha_n(b); a) = 1\). So, \((\alpha_0(b); a) = 1\) or \((\alpha_n(b); a) = 1\) for \(n \in \mathbb{N}\). By (4), \(b \leq a\) or \((\alpha_1(b); a) = 1\). If \(b \leq a\), as \(a \leq \Box b \rightarrow a\), result that \(b \leq \Box b \rightarrow a\) and so, \((\alpha_1(b); a) = 1\). The converse is an immediate consequence of Theorem 6.9. \(\Box\)

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References


Sergio A. Celani  
CONICET and Departamento de Matemáticas,  
Universidad Nacional del Centro,  
Pinto 399; 7000 Tandil, Argentina  
scelani@exa.unicen.edu.ar

Daniela Montangie  
Universidad Nacional del Comahue,  
Facultad de Economía y Administración,  
Departamento de Matemática  
Buenos Aires 1400; 8300 Neuquén, Argentina  
dmontang@gmail.com