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# AN INVESTIGATION OF THE APPROXIMATION OF FUNCTIONS OF TWO VARIABLES BY THE POISSON INTEGRAL FOR HERMITE EXPANSIONS

# APROKSYMACJA FUNKCJI DWÓCH ZMIENNYCH CAŁKĄ POISSONA ZWIĄZANĄ Z WIELOMIANAMI HERMITE'A

A b s t r a c t

This paper presents a study of the approximation properties of the Poisson integral for Hermite expansions in the space $L^p$. The rate of convergence of functions of two variables by these integrals is established.

*Keywords: rate of convergence; Poisson integral; Hermite expansions, positive linear operators*

S t r e s z c z e n i e

Artykuł poświęcony jest własnościom aproksymacyjnym całek Poissona związanych z wielomianami Hermite'a. Udowodniono twierdzenie o rzędzie zbieżności funkcji dwóch zmiennych w przestrzeni $L^p$ tymi operatorami.

*Słowa kluczowe: promień zbieżności, całka Poissona, wielomiany Hermite'a, dodatnie operatory liniowe*

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## 1. Introduction

Let $1 \le p \le \infty$, we denote by $L^p(\mathbb{R}^2)$ the set of all the Lebesgue measurable functions $f$ defined on $\mathbb{R}^2$ such that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(t_1,t_2)|^p \, dt_1\, dt_2 < \infty$ if $1 \le p < \infty$, and if $p = \infty$ we require $f$ to be bounded almost everywhere on $\mathbb{R}^2$.

In this paper, we present approximation properties of the Poisson integral $\overline{A}$ in the space $L^p(\mathbb{R}^2)$, $1 \le p \le \infty$ defined by:

$$\overline{A}(f;r,y_1,y_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} rK(r,y_1,z_1)K(r,y_2,z_2)f(z_1,z_2)dz_1\, dz_2, \quad 0 < r < 1,$$

where:

$$K(r,y,z) = \sum_{n=0}^{\infty} r^n h_n(y)h_n(z) = \frac{1}{\left(\pi(1-r^2)\right)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\cdot\frac{1+r^2}{1-r^2}\left(y^2+z^2\right)+\frac{2ryz}{1-r^2}\right),$$

$$h_n(x) = \left(2^n n!\sqrt{\pi}\right)^{-\frac{1}{2}} \exp\left(-\frac{x^2}{2}\right) H_n(x)$$

and $H_n$ is the $n$ th Hermite polynomial (see [11]). The norm in $L^p(\mathbb{R}^2)$ is given by:

$$\| f \|_p = \begin{cases} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(t_1,t_2)|^p \, dt_1\, dt_2\right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \sup\limits_{(t_1,t_2)\in\mathbb{R}^2} ess\, |f(t_1,t_2)|, & p = \infty. \end{cases}$$

Some convergence theorems, the Voronovskaya formula, and a boundary value problem for the integral $\overline{A}$ were presented in [5]. The following result was proved (see [5]):
Theorem 1 Let $\overline{y} = (\overline{y}_1, \overline{y}_2) \in \mathbb{R}^2$ and $f = f_1 + f_2$, where $f_1 \in L^1(\mathbb{R}^2)$, $f_2 \in L^\infty(\mathbb{R}^2)$. If $f$ is continuous at $\overline{y}$, then:

$$\lim_{(r,y)\to(1^-,\overline{y})} \overline{A}(f;r,y) = f(\overline{y}), \quad y = (y_1, y_2).$$

In this paper we shall give an order of approximation of functions belonging to $L^p(\mathbb{R}^2)$ by the operator $\overline{A}$. It is worth mentioning that approximation properties of Poisson integrals for orthogonal expansions and their various modifications were also studied in [4, 12, 6–10], in one and two dimensions.

Some auxiliary results, which will be needed in the next part of this paper, are now presented. It is clear that $\overline{A}(f;r,y_1,y_2) = rA(f_1;r,y_1)A(f_2;r,y_2)$ for $f_1, f_2 \in L^p(\mathbb{R})$ and such that $f(z_1,z_2) = f_1(z_1)f_2(z_2)$, where $A(f)(r,y) = A(f;r,y) = \int_{-\infty}^{\infty} K(r,y,z)f(z)dz, \quad 0 < r < 1.$

The operator $\overline{A}$ is linear and positive. Basic facts on positive linear operators and its applications can be found in [2, 3].

In paper [7], we can find the following equalities:

$$A(1;r,y)=\left(\frac{2}{1+r^2}\right)^{1/2}\exp\left(-\frac{1}{2}\cdot\frac{1-r^2}{1+r^2}y^2\right),$$

$$A(\phi_{m,y};r,y)=\left(\frac{2}{1+r^2}\right)^{1/2}\exp\left(-\frac{1}{2}\cdot\frac{1-r^2}{1+r^2}y^2\right)$$

$$\times\sum_{p=0}^{\left[\frac{m}{2}\right]}\binom{m}{p}\frac{(m-p)!}{(m-2p)!2^p}\cdot\left(\frac{1-r^2}{1+r^2}\right)^p\left(-\frac{(1-r)^2}{1+r^2}y\right)^{m-2p}$$

for $0<r<1$, $y\in R$, where $[a]$ denotes the integral part of $a\in R$ and $\varphi_{m,y}(z)=(z-y)^m$.

From the above, we have the following result in the bivariate case.

**Lemma 1** *Let* $\varphi_{n,y_i}(z_1,z_2)=(z_i-y_i)^n$, $y_i,z_i\in R$, $i=1,2$, $n\in N$. *It holds*

$$\overline{A}\left(\left|\varphi_{1,y_i}\right|;r,y_1,y_2\right)\le\frac{2r}{1+r^2}\cdot\left(\frac{1-r^2}{1+r^2}+\frac{(r-1)^4}{(1+r^2)^2}y_i^2\right)^{\frac{1}{2}}\exp\left(-\frac{1}{2}\cdot\frac{1-r^2}{1+r^2}\left(y_1^2+y_2^2\right)\right) \qquad (1)$$

for $0<r<1$.

Proof. Using Hölder's inequality, we get:

$$\overline{A}\left(\left|\phi_{1,y_1}\right|;r,y_1,y_2\right)\le\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}rK(r,y_1,z_1)K(r,y_2,z_2)dz_1dz_2\right)^{\frac{1}{2}}$$

$$\times\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}rK(r,y_1,z_1)K(r,y_2,z_2)\,|\,z_1-y_1\,|^2\,dz_1dz_2\right)^{\frac{1}{2}}=\left(\overline{A}(1;r,y_1,y_2)\right)^{\frac{1}{2}}\cdot\left(\overline{A}\left(\phi_{2,y_1};r,y_1,y_2\right)\right)^{\frac{1}{2}} \qquad (2)$$

for $(y_1,y_2)\in R^2$ and $0<r<1$. We have (see [5]):

$$\overline{A}(1;r,y_1,y_2)=\frac{2r}{1+r^2}\exp\left(-\frac{1}{2}\cdot\frac{1-r^2}{1+r^2}\left(y_1^2+y_2^2\right)\right),$$

$$\overline{A}\left(\phi_{2,y_i};r,y_1,y_2\right)=\frac{2r}{1+r^2}\left(\frac{1-r^2}{1+r^2}+\frac{(r-1)^4}{\left(1+r^2\right)^2}y_i^2\right)\exp\left(-\frac{1}{2}\cdot\frac{1-r^2}{1+r^2}\left(y_1^2+y_2^2\right)\right),\ i=1,2.$$

From this and (2) we obtain (1) for $i=1$. Analogously, we calculate (1) for $i=2$.

## 2. Rate of convergence

In this section, we give an order of approximation of function of two variables in the space $L^p$.

We achieve this using the modulus of continuity $\omega(f;\delta_1,\delta_2)$, $\delta_1,\delta_2>0$ of $f\in L^p(R^2)$ defined as follows:

$$\omega(f;\delta_1,\delta_2)=\sup_{\substack{0<h_1\le\delta_1\\0<h_2\le\delta_2}}\left\{\sup_{(y_1,y_2)\in\mathrm{R}^2}\left|f(y_1+h_1,y_2+h_2)-f(y_1,y_2)\right|\right\}.$$

First, we prove the following lemma, which we will use in the proof of the approximation theorem.

We shall apply the method used in [12].

**Lemma 2** *Let* $f\in C^1(R^2)\cap L^p(R^2)$, $1\le p\le\infty$. *Therefore*

$$\left|\overline{A}(f;r,y_1,y_2)-f(y_1,y_2)\overline{A}(1;r,y_1,y_2)\right|\le\frac{2r}{1+r^2}\exp\left(-\frac{1}{2}\cdot\frac{1-r^2}{1+r^2}\left(y_1^2+y_2^2\right)\right)$$

$$\times\left\{\left(\frac{1-r^2}{1+r^2}+\frac{(r-1)^4y_1^2}{(1+r^2)^2}\right)^{\frac{1}{2}}\sup_{(y_1,y_2)\in R^2}\left|\frac{\partial f(y_1,y_2)}{\partial y_1}\right|+\left(\frac{1-r^2}{1+r^2}+\frac{(r-1)^4y_2^2}{(1+r^2)^2}\right)^{\frac{1}{2}}\sup_{(y_1,y_2)\in R^2}\left|\frac{\partial f(y_1,y_2)}{\partial y_2}\right|\right\}.$$

for $0<r<1$ and all $(y_1,y_2)\in R^2$.

Proof. Let $(y_1,y_2)\in R^2$. be a fixed point and $f\in C^1(R^2)\cap L^p(R^2)$. We have:

$$f(z_1,z_2)-f(y_1,y_2)=\int_{y_1}^{z_1}\frac{\partial}{\partial u}f(u,z_2)du+\int_{y_2}^{z_2}\frac{\partial}{\partial v}f(y_1,v)dv$$

for all $(z_1,z_2)\in R^2$. Let us denote:

$$\lambda_{y_1}(z_1,z_2)=\int_{y_1}^{z_1}\frac{\partial}{\partial u}f(u,z_2)du,\quad\tau_{y_2}(z_1,z_2)=\int_{y_2}^{z_2}\frac{\partial}{\partial v}f(y_1,v)dv.$$

Observe that:

$$\left|\lambda_{y_1}(z_1,z_2)\right|\le|z_1-y_1|\sup_{(y_1,y_2)\in\mathrm{R}^2}\left|\frac{\partial f(y_1,y_2)}{\partial y_1}\right|,\ \left|\tau_{y_2}(z_1,z_2)\right|\le|z_2-y_2|\sup_{(y_1,y_2)\in\mathrm{R}^2}\left|\frac{\partial f(y_1,y_2)}{\partial y_2}\right|.\qquad(3)$$

From (3) and Lemma 1, we obtain:

$$\overline{A}\left(\left|\lambda_{y_1}\right|;r,y_1,y_2\right)\le\overline{A}\left(\left|\varphi_{1,y_1}\right|;r,y_1,y_2\right)\sup_{(y_1,y_2)\in\mathrm{R}^2}\left|\frac{\partial f(y_1,y_2)}{\partial y_1}\right|$$

$$\leq \frac{2r}{1+r^2}\left(\frac{1-r^2}{1+r^2}+\frac{(r-1)^4 y_1^2}{(1+r^2)^2}\right)^{\frac{1}{2}} \exp\left(-\frac{1}{2}\cdot\frac{1-r^2}{1+r^2}\left(y_1^2+y_2^2\right)\right) \sup_{(y_1,y_2)\in \mathrm{R}^2}\left|\frac{\partial f(y_1,y_2)}{\partial y_1}\right|,$$

$$\overline{A}\left(\left|\tau_{y_2}\right|;r,y_1,y_2\right) \quad \leq \frac{2r}{1+r^2}\left(\frac{1-r^2}{1+r^2}+\frac{(r-1)^4 y_2^2}{(1+r^2)^2}\right)^{\frac{1}{2}} \exp\left(-\frac{1}{2}\cdot\frac{1-r^2}{1+r^2}\left(y_1^2+y_2^2\right)\right)$$

$$\sup_{(y_1,y_2)\in \mathrm{R}^2}\left|\frac{\partial f(y_1,y_2)}{\partial y_2}\right|.$$

Hence:

$$\left|\overline{A}\left(f;r,y_1,y_2\right)-f(y_1,y_2)\overline{A}\left(1;r,y_1,y_2\right)\right| \quad \leq \frac{2r}{1+r^2}\exp\left(-\frac{1}{2}\cdot\frac{1-r^2}{1+r^2}\left(y_1^2+y_2^2\right)\right)$$

$$\times\left\{\left(\frac{1-r^2}{1+r^2}+\frac{(r-1)^4 y_1^2}{(1+r^2)^2}\right)^{\frac{1}{2}} \sup_{(y_1,y_2)\in \mathrm{R}^2}\left|\frac{\partial f(y_1,y_2)}{\partial y_1}\right| + \left(\frac{1-r^2}{1+r^2}+\frac{(r-1)^4 y_2^2}{(1+r^2)^2}\right)^{\frac{1}{2}} \sup_{(y_1,y_2)\in \mathrm{R}^2}\left|\frac{\partial f(y_1,y_2)}{\partial y_2}\right|\right\}$$

and the proof of the lemma is completed.

We are now in a position to prove the approximation theorem.

**Theorem 2** *Let* $f \in C(R^2)\cap L^p(R^2),\ 1 \leq p \leq \infty$. *Therefore*

$$\left|\overline{A}(f;r,y_1,y_2)-f(y_1,y_2)\overline{A}(1;r,y_1,y_2)\right|$$

$$\leq 6\omega\left(f;\sqrt{\frac{1-r^2}{1+r^2}+\frac{(r-1)^4}{(1+r^2)^2}y_1^2},\sqrt{\frac{1-r^2}{1+r^2}+\frac{(r-1)^4}{(1+r^2)^2}y_2^2}\right)$$

for $0 < r < 1$ and all $(y_1, y_2)\in \mathrm{R}^2$.

Proof. Let $(y_1, y_2)\in \mathrm{R}^2$ be a fixed point and $f_{\delta_1,\delta_2}$ be the Steklov mean defined by:

$$f_{\delta_1,\delta_2}(y_1,y_2)=\frac{1}{\delta_1\delta_2}\int_0^{\delta_1}\int_0^{\delta_2} f(y_1+u,y_2+v)\,dudv \quad \text{for} \quad (y_1,y_2)\in R^2,\delta_1,\delta_2>0.$$

From this definition, we conclude that:

$$f_{\delta_1,\delta_2}(y_1,y_2)-f(y_1,y_2)=\frac{1}{\delta_1\delta_2}\int_0^{\delta_1}\int_0^{\delta_2}\Delta_{u,v}f(y_1,y_2)\,dudv,$$

$$\frac{\partial}{\partial y_1}f_{\delta_1,\delta_2}(y_1,y_2)=\frac{1}{\delta_1\delta_2}\int_0^{\delta_2}\left(\Delta_{\delta_1,v}f(y_1,y_2)-\Delta_{0,v}f(y_1,y_2)\right)dv,$$

$$\frac{\partial}{\partial y_2} f_{\delta_1,\delta_2}(y_1,y_2)=\frac{1}{\delta_1\delta_2}\int_0^{\delta_1}\Big(\Delta_{u,\delta_2} f(y_1,y_2)-\Delta_{u,0} f(y_1,y_2)\Big)du,$$

where

$$\Delta_{u,v} f(y_1,y_2)=f(y_1+u,y_2+v)-f(y_1,y_2).$$

Hence, if $f\in C(R^2)\cap L^p(R^2)$, then $f_{\delta_1,\delta_2}\in C^1(R^2)\cap L^p(R^2)$. Moreover

$$\sup_{(y_1,y_2)\in R^2}\left|f_{\delta_1,\delta_2}(y_1,y_2)-f(y_1,y_2)\right|\le \omega(f;\delta_1,\delta_2),$$

$$\sup_{(y_1,y_2)\in R^2}\left|\frac{\partial}{\partial y_1} f_{\delta_1,\delta_2}(y_1,y_2)\right|\le 2\delta_1^{-1}\omega(f;\delta_1,\delta_2), \qquad (4)$$

for all $\delta_1,\delta_2>0$. Observe that

$$\left|\overline{A}(f;r,y_1,y_2)-f(y_1,y_2)\overline{A}(1;r,y_1,y_2)\right|$$

$$\le\left|\overline{A}(f-f_{\delta_1,\delta_2};r,y_1,y_2)\right|+\left|\overline{A}(f_{\delta_1,\delta_2};r,y_1,y_2)-f_{\delta_1,\delta_2}(y_1,y_2)\overline{A}(1;r,y_1,y_2)\right|$$

$$+\left|f_{\delta_1,\delta_2}(y_1,y_2)-f(y_1,y_2)\right|\cdot\overline{A}(1;r,y_1,y_2),\ \ (y_1,y_2)\in R^2,\delta_1,\delta_2>0.$$

From Lemma 2 and (4) we obtain

$$\left|\overline{A}(f_{\delta_1,\delta_2};r,y_1,y_2)-f_{\delta_1,\delta_2}(y_1,y_2)\overline{A}(1;r,y_1,y_2)\right|$$

$$\le\frac{2r}{1+r^2}\exp\left(-\frac{1}{2}\cdot\frac{1-r^2}{1+r^2}\left(y_1^2+y_2^2\right)\right)\left\{2\delta_1^{-1}\omega(f;\delta_1,\delta_2)\left(\frac{1-r^2}{1+r^2}+\frac{(r-1)^4y_1^2}{(1+r^2)^2}\right)^{\frac{1}{2}}\right.$$

$$\left.+2\delta_2^{-1}\omega(f;\delta_1,\delta_2)\left(\frac{1-r^2}{1+r^2}+\frac{(r-1)^4y_2^2}{(1+r^2)^2}\right)^{\frac{1}{2}}\right\}$$

$$\le 2\omega(f;\delta_1,\delta_2)\left\{\delta_1^{-1}\left(\frac{1-r^2}{1+r^2}+\frac{(r-1)^4y_1^2}{(1+r^2)^2}\right)^{\frac{1}{2}}+\delta_2^{-1}\left(\frac{1-r^2}{1+r^2}+\frac{(r-1)^4y_2^2}{(1+r^2)^2}\right)^{\frac{1}{2}}\right\}.$$

Using (4) we have:

$$\left|f_{\delta_1,\delta_2}(y_1,y_2)-f(y_1,y_2)\right|\cdot \overline{A}(1;r,y_1,y_2)\le \overline{A}(1;r,y_1,y_2)\omega(f;\delta_1,\delta_2)\le \omega(f;\delta_1,\delta_2)$$

and

$$\left|\overline{A}(f-f_{\delta_1,\delta_2};r,y_1,y_2)\right| \le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} rK(r,y_1,z_1)K(r,y_2,z_2)\,|\,f(z_1,z_2)-f_{\delta_1,\delta_2}(z_1,z_2)\,|dz_1\,dz_2$$

$$\le \sup_{(y_1,y_2)\in \mathsf{R}^2}\left|f_{\delta_1,\delta_2}(y_1,y_2)-f(y_1,y_2)\right|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} rK(r,y_1,z_1)K(r,y_2,z_2)dz_1\,dz_2$$

$$\le \overline{A}(1;r,y_1,y_2)\omega(f;\delta_1,\delta_2)\le \omega(f;\delta_1,\delta_2).$$

Finally, we get:

$$\left|\overline{A}(f;r,y_1,y_2)-f(y_1,y_2)\overline{A}(1;r,y_1,y_2)\right|$$

$$\le 2\omega(f;\delta_1,\delta_2)\left\{1+\delta_1^{-1}\left(\frac{1-r^2}{1+r^2}+\frac{(r-1)^4}{(1+r^2)^2}y_1^2\right)^{\frac{1}{2}}+\delta_2^{-1}\left(\frac{1-r^2}{1+r^2}+\frac{(r-1)^4}{(1+r^2)^2}y_2^2\right)^{\frac{1}{2}}\right\}$$

for all $(y_1,y_2)\in R^2,\delta_1,\delta_2>0.$ Choosing:

$$\delta_1=\left(\frac{1-r^2}{1+r^2}+\frac{(r-1)^4}{(1+r^2)^2}y_1^2\right)^{\frac{1}{2}},\quad \delta_2=\left(\frac{1-r^2}{1+r^2}+\frac{(r-1)^4}{(1+r^2)^2}y_2^2\right)^{\frac{1}{2}},$$

we obtain the desired estimation for $\overline{A}$.
From Theorem 2, we can derive the following result.

**Corollary 1** *Let* $f\in C(R^2)\cap L^p(R^2),\ 1\le p\le\infty$. *Then it holds*

$$\left|\overline{A}(f;r,y_1,y_2)-f(y_1,y_2)\right|\le 6\omega\left(f;\sqrt{\frac{1-r^2}{1+r^2}+\frac{(r-1)^4}{(1+r^2)^2}y_1^2},\sqrt{\frac{1-r^2}{1+r^2}+\frac{(r-1)^4}{(1+r^2)^2}y_2^2}\right)$$

$$+|\,f(y_1,y_2)\,|\cdot\left|\overline{A}(1;r,y_1,y_2)-1\right|$$

for $0<r<1$ and all $(y_1,y_2)\in \mathsf{R}^2$.

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