TECHNICAL TRANSACTIONS | CZASOPISMO TECHNICZNE
FUNDAMENTAL SCIENCES | NAUKI PODSTAWOWE
2-NP/2014

LUDWIK BYSZEWSKI AND TERESA WINIARSKA*

THE EXISTENCE AND UNIQUENESS OF SOLUTIONS OF THE DIRICHLET NONLOCAL PROBLEM WITH A NONLOCAL INITIAL CONDITION

ISTNIENIE I JEDNOZNACZNOŚĆ ROZWIĄZAŃ NIELOKALNEGO ZAGADNIENIA DIRICHLETA Z NIELOKALNYM WRUNKIEM POCZĄTKOWYM

A b s t r a c t

The aim of this paper is to prove the existence and uniqueness of solutions of the Dirichlet nonlocal problem with nonlocal initial condition. The considerations are extensions of results by E. Andreu-Vaillo, J. M. Mazón, J. D. Rossi and J. J. Toledo-Melero [1].

Keywords:existence and uniqueness of solutions, Dirichlet problem, nonlocal problem, nonlocal initial condition

S t r e s z c z e n i e

W artykule udowodniono istnienie i jednoznaczność rozwiązań nielokalnego zagadnienia Dirichleta z nielokalnym warunkiem początkowym. Rozważania są rozszerzeniami rezultatów otrzymanych przez E. Andreu-Vaillo, J. M. Mazóna, J. D. Rossi i J. J. Toledo-Melero [1].

Słowa kluczowe: istnienie i jednoznaczność rozwiązań, zagadnienie Dirichleta, zagadnienie nielokalne, nielokalny warunek początkowy

*Institute of Mathematics, Cracow University of Technology, Poland; lbyszews@usk.pk.edu.pl, twiniars@usk.pk.edu.pl

1. Preliminaries

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Moreover, let T be a fixed positive number and $k \in \mathbb{R} \setminus \{0\}$.

We will need the following assumption:

Assumption (H) (see: [1]). *$J \in C(\mathbb{R}^n, \mathbb{R})$ is a nonnegative radial function with $J(0) > 0$ and*

$$\int_{\mathbb{R}^n} J(x)dx = 1.$$

In [1], the existence and uniqueness of a solution of the following nonlocal Dirichlet boundary value problem

$$\begin{cases} u_t(x,t) = \int_{\mathbb{R}^n} J(x-y)\big(u(y,t) - u(x,t)\big)dy, & x \in \Omega, \quad t > 0, \\ u(x,t) = g(x,t), \quad x \notin \Omega, \quad t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega \end{cases}$$

is studied.

For this purpose the Banach fixed point theorem is applied in [1].

The existence and uniqueness of solutions of differential problems were, also, studied using the Banach fixed point theorem, by Kamont [2], Muszyński and Myszkis [3], and Pelczar and Szarski [4].

The aim of the paper is to give a theorem on the existence and uniqueness of a solution of the following nonlocal Dirichlet boundary value problem together with the nonlocal initial condition

$$\begin{cases} u_t(x,t) = \int_{\mathbb{R}^n} J(x-y)\big(u(y,t) - u(x,t)\big)dy, & x \in \Omega, \quad t \in (0,T), \\ u(x,t) = g(x,t), \quad x \notin \Omega, \quad t \in (0,T), \\ u(x,0) + kTu(x,T) = u_0(x), \; x \in \Omega. \end{cases} \tag{1.1}$$

For this purpose we will also apply the Banach fixed point theorem.

We will need the assumption:

Assumption (F). $u_0 \in L^1(\Omega)$ and $g \in C\big((0,T); L^1(\mathbb{R}^n \backslash \Omega)\big)$.

2. Existence and uniqueness of Solutions

Let Assumptions (H) and (F) be satisfied in this section.

Definition 2.1. *A function $u \in C\Big([0,T]; L^1(\mathbb{R}^n)\Big)$ is said to be a solution of nonlocal problem (1.1) if*

$$u(x,t) = u_0(x) - kTu(x,T)$$
$$+ \int_0^t \int_{\mathbb{R}^n} J(x-y)\big(u(y,s) - u(x,s)\big)dyds, \quad x \in \Omega,\ t \in (0,T),$$

and

$$u(x,t) = g(x,t) \ \ for \ \ x \notin \Omega, \ \ t \in (0,T).$$

Consider the Banach space

$$X_T = \big\{ w \in C([0,T]; L^1(\Omega)) \big\}$$

with the norm

$$|||w||| = \max_{0 \le t \le T} \parallel w(\cdot,t) \parallel_{L^1(\Omega)} .$$

The solution of problem (1.1) will be obtained as a fixed point of the operator

$$\mathcal{T}_{w_0} : X_T \longrightarrow X_T$$

defined by the formula

$$\mathcal{T}_{w_0}(w)(x,t) = w_0(x) - kTw(x,T)$$
$$+ \int_0^t \int_{\mathbb{R}_n} J(x-y)\big(w(y,s) - w(x,s)\big)dyds, \ \ x \in \Omega, \ \ t \in (0,T),$$

where

$$w(x,t) = g(x,t) \ \ \text{for} \ \ x \notin \Omega, \ \ t \in (0,T).$$

To prove the existence and uniqueness of the solution of problem (1.1), we will need the following lemma:

Lemma 2.1. *Let $w_0, z_0 \in L^1(\Omega)$. Then there is a constant*

$$C = \mid k \mid + \tilde{k}, \ \ where \ \ \tilde{k} > 0, \tag{2.2}$$

depending on J and Ω such that

$$|||\mathcal{T}_{w_0}(w) - \mathcal{T}_{z_0}(z)||| \le \parallel w_0 - z_0 \parallel_{L^1(\Omega)} + CT|||w - z|||$$

for all $w, z \in X_T$.

Proof. Observe that

$$\int_\Omega \mid \mathcal{T}_{w_0}(w)(x,t) - \mathcal{T}_{z_0}(z)(x,t) \mid dx \le$$

$$\leq \int_{\Omega} \mid w_0 - z_0 \mid (x)dx + \mid k \mid T \int_{\Omega} \mid w(x,T) - z(x,T) \mid dx$$

$$+ \int_{\Omega} \mid \int_0^t \int_{\mathbb{R}^n} J(x-y)[(w(y,s) - z(y,s)) - \big(w(x,s) - z(x,s)\big)]dyds \mid dx$$

$$\leq \| w_0 - z_0 \|_{L^1(\Omega)} + \mid k \mid T|||w - z||| + \tilde{k}T|||w - z|||$$

$$= \| w_0 - z_0 \|_{L^1(\Omega)} + (\mid k \mid + \tilde{k})T|||w - z|||, \ w, z \in X_T,$$

where $\tilde{k}$ is a positive constant depending on J and Ω.

Consequently, since $w - z$ vanishes outside of Ω then

$$|||\mathcal{T}_{w_0}(w) - \mathcal{T}_{z_0}(z)||| \leq$$

$$\leq \| w_0 - z_0 \|_{L^1(\Omega)} + CT|||w - z||| \quad \text{for} \quad w, z \in X_T.$$

The proof of Lemma 2.1 is complete.

Applying Lemma 2.1 we will prove the existence and uniqueness of the solution of problem (1.1).

Theorem 2.1 *Let Assumptions* (H) *and* (F) *be satisfied. Moreover, let* $CT < 1$, *where* C *is given by (2.2).*

Then there is a unique solution of problem (1.1) on the interval $[0, T]$.

Proof. Firstly, we will show that $\mathcal{T}_{u_0}$ maps X_T into X_T. Let $z_0 \equiv 0$, $z \equiv 0$ and $w_0 \equiv u_0$ in Lemma 2.1. Then

$$\mathcal{T}_{u_0}(w) \in C\big([0, T]; L^1(\Omega)\big)$$

for $w \in X_T$.

Since $CT < 1$ then taking $z_0 \equiv w_0 \equiv u_0$ in Lemma 2.1 we get that $\mathcal{T}_{u_0}$ is a strict contraction in X_T and the existence and uniqueness of the solution of problem (1.1) follows from the Banach fixed point theorem on the interval $[0, T]$.

The proof of Theorem 2.1 is complete.

References

[1] Andreu-Vaillo F., Mazón J. M., Rossi J. D., Toledo-Melero J. J., *Nonlocal Diffusion Problems,* American Mathematical Society, Providence, Rhode Island 2010.

[2] Kamont Z., *Ordinary Differential Equations,* Wydawnictwo Uniwersytetu Gdańskiego, Gdańsk 1999 [in Polish].

[3] Muszyński J., Myszkis A. D., *Ordinary Differential Equations,* Państwowe Wydawnictwo Naukowe, Warszawa 1984 [in Polish].

[4] Pelczar A., Szarski J., *Introduction to the Theory of Differential Equations,* Państwowe Wydawnictwo Naukowe, Warszawa 1987 [in Polish].