FIXED POINTS OF $\alpha$-NONEXPANSIVE MAPPINGS

Abstract
This paper is connected with the theory of $\alpha$-nonexpansive mappings, which were introduced by K. Goebel and M. A. J. Pineda in 2007. These mappings are a natural generalisation of nonexpansive mappings from the point of view of the fixed point theory. In particular, they proved that in Banach spaces all $\alpha = (\alpha_1, \ldots, \alpha_n)$ -nonexpansive mappings with $\alpha_i$ big enough, namely $\alpha_1 \geq 2^{1-n}$, have minimal displacement equal to zero. This paper introduces some new results connected with this problem.

Keywords: $\alpha$-nonexpansive mappings, minimal displacement, fixed point

Streszczenie
Niniejszy artykuł jest związany z odwzorowaniami $\alpha$-nieoddalającymi, które zostały wprowadzone przez K. Goebla i M. A. J. Pinedę w 2007 r. Odwzorowania te są naturalnym uogólnieniem odwzorowań nieoddalających z punktu widzenia teorii punktu stałego. Wyżej wspomniani autorzy wykazali, że w przestrzeniach Banacha odwzorowania $\alpha = (\alpha_1, \ldots, \alpha_n)$ -nieoddalające, mające odpowiednio duże $\alpha_i$, a dokładnie $\alpha_1 \geq 2^{1-n}$, posiadają minimalne przesunięcie równe zeru. W artykule przedstawiono pewne nowe wyniki z związane z tym problemem.

Słowa kluczowe: odwzorowania $\alpha$-nieoddalające, minimalne przesunięcie, punkt stały

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1. Introduction and preliminaries

Let \((X, d)\) be a metric space, and let \(\alpha = (\alpha_1, \ldots, \alpha_n)\) be a multi-index satisfying \(\alpha_1 > 0, \alpha_n > 0, \alpha_i \geq 0, i = 2, \ldots, n-1\) and \(\sum_{i=1}^{n} \alpha_i = 1\). In [2], the following notions were introduced:

The mapping \(T : X \to X\) is said to be \(\alpha\)-Lipschitzian with constant \(k \geq 0\), if

\[
\sum_{i=1}^{n} \alpha_i \cdot d(T^{i-1}x, T^{i-1}y) \leq kd(x, y) \quad \text{for all } x, y \in X.
\]

The mapping \(T : X \to X\) is said to be \(\alpha\)-nonexpansive (\(\alpha\)-contraction), if \(T\) is \(\alpha\)-Lipschitzian with constant \(k = 1\) (\(k < 1\) resp.).

Denote the Lipschitz constant with \(k(T)\) and the \(\alpha\)-Lipschitz constant of \(T\) with \(k(\alpha, T)\).

Define also \(d(T) := \inf\{d(x, Tx), x \in X\}\), which we will call the minimal displacement of \(T\). Sometimes it is also called the approximate fixed point of \(T\).

These notions are natural generalisations of Lipschitzian mappings, nonexpansive mappings and contractions from the point of view of the fixed point theory. For more information concerning \(\alpha\)-nonexpansive mappings and other Lipschitzian mappings connected with the fixed point theory, we refer to [4].

In [2], the authors proved the following:

**Theorem 1.1.** (see also [4], chapter 3) Let \(X\) be a Banach space, let \(C\) be a nonempty, closed, convex and bounded subset of \(X\). Let \(T : C \to C\) be an \(\alpha = (\alpha_1, \ldots, \alpha_n)\)-nonexpansive mapping where \(\alpha_1 \geq \frac{1}{2^{1-n}}\). Then \(d(T) = 0\).

Notice that the problem of determining the set of multi-indices \(\alpha\) for which each \(\alpha\)-nonexpansive mapping \(T\) has \(d(T) = 0\) is still open.

The aim of this paper is to prove two results (Theorem 2.1, Theorem 2.2) which give a partial answer to the above open problem (see [4]).

Before proceeding further, let us recall the generalised Banach contraction principle (abbreviated to GBCP), which is formulated as follows:

**Theorem 1.2.** ([1], [5]) In complete metric space \(X\) if for some \(N \geq 1\) and \(0 < M < 1\) the mapping \(T : X \to X\) satisfies

\[
\min\{d(T^{j-1}x, T^{j-1}y), 1 \leq j \leq N\} \leq Md(x, y) \quad \text{for any } x, y \in X,
\]

then \(T\) has the unique fixed point.

In author’s PhD thesis [6] the more general version of the above theorem was presented. Let us recall it without proof.

**Theorem 1.3.** Let \((X, d)\) be a complete metric space, \(N \geq 1\). Assume that \(\phi : [0, \infty) \to [0, 1]\) is a continuous, non-increasing function satisfying \(\phi(t) = 1\) if, and only if, \(t = 0\). Let \(T : X \to X\) be such that

\[
\min\{d(T^{j-1}x, T^{j-1}y), 1 \leq j \leq N\} \leq \phi(d(x, y)) \cdot d(x, y) \quad \text{for all } x, y \in X.
\]

Then \(T\) has the unique fixed point.
2. Main results

Firstly, let us note a simple fact, there exist some $\alpha$-Lipschitzian mappings which are not $\alpha$-nonexpansive; however, their minimal displacement is equal to zero; moreover, they may have the unique fixed point.

This is illustrated by:

**Example 2.1.** Let $T: l_\infty \cap \{x \in l_\infty : x_i \geq 0, \ i \in \mathbb{N}\} \to l_\infty \cap \{x \in l_\infty : x_i \geq 0, \ i \in \mathbb{N}\}$ be defined in the following way: $T: x = (x_1, x_2, \ldots) \to Tx := \left(1, \frac{2x_3}{1+x_3}, \frac{1}{2}x_2, \frac{2x_4}{1+x_5}, \frac{1}{2}x_4, \ldots\right)$. Then $T$ is not $\alpha$-nonexpansive for any $\alpha$; however, for properly chosen $\alpha = (\alpha_1, \alpha_2)$ the mapping $T$ is $\alpha$-Lipschitzian with constant $k$ arbitrarily close to 1. Moreover, $T$ has the unique fixed point.

Obviously, the mapping $T$ has the unique fixed point $(1, 0, 0, \ldots)$.

Also, we have $\|Tx - Ty\| \leq 2\|x - y\|$ and $\|T^i x - T^i y\| \leq \|x - y\|, \ i \geq 2$ for any $x, y \in l_\infty \cap \{x \in l_\infty : x_i \geq 0, \ i \in \mathbb{N}\}$. On the other hand, taking $x^n = \left(0, 0, \frac{1}{n}, 0, 0, \ldots\right)$ and $y^n = (0, 0, \ldots)$ we have $\frac{\|Tx^n - Ty^n\|}{\|x^n - y^n\|} = \frac{2 \cdot \frac{1}{n}}{1+\frac{1}{n}} = \frac{2}{n+1} \to 2$, $n \to \infty$; therefore, $k(T) = 2$.

Similarly, $k(T^2) = 1$ and $k(T^i) \geq 1, \ i \geq 3$.

In $l_\infty$, it is not possible to choose $\alpha$ such that $\alpha_1 > 0$ and $T$ is $\alpha$-nonexpansive; however, $\frac{1}{n} \|Tx - Ty\| + \frac{n-1}{n} \|T^2 x - T^2 y\| \leq \frac{n+1}{n} \|x - y\|$; therefore, assuming $n$ to be big enough, the mapping $T$ is $\alpha = \left(\frac{1}{n}, \frac{n-1}{n}\right)$-Lipschitzian with constant $k(\alpha, T)$ arbitrarily close to 1.

It is worth mentioning that the existence and uniqueness of the fixed point of $T$ also follows from Theorem 1.3. Indeed, we have $T^2 x = \left(1, \frac{x_2}{1+\frac{3}{2}x_2}, \frac{x_3}{1+3x_3}, \ldots\right)$. 
\[
\begin{align*}
\left(\frac{x_4}{1+\frac{3}{2}x_4}, \frac{x_5}{1+3x_5}, \ldots\right) \quad \text{and} \quad \left|\frac{x_i - y_i}{1+\frac{3}{2}x_i} - \frac{y_i}{1+\frac{3}{2}y_i}\right| & \leq \left|\frac{x_i - y_i}{1+\frac{3}{2}(x_i + y_i) + \frac{9}{4}x_iy_i}\right| \\
\leq \frac{|x_i - y_i|}{1+|x_i + y_i|} & \leq \frac{\|x - y\|}{1+\|x - y\|}, \quad i \in \mathbb{N}
\end{align*}
\]

The latter inequality follows from the fact that \( t \to \frac{t}{1+t} \) is an increasing function on \([0, \infty)\). Similarly, \( \frac{x_i}{1+3x_i} - \frac{y_i}{1+3y_i} \leq \frac{\|x - y\|}{1+\|x - y\|} \); therefore,

\[
\|T^2x - T^2y\| \leq \frac{1}{1+\|x - y\|}\|x - y\|, \quad \text{so } T \text{ satisfies the assumptions of Theorem 1.3 with } \\
\phi(t) := \frac{1}{1+t}.
\]

Now, let us exchange the condition \( \alpha_1 \geq 2^{1-n} \) with the other regularity condition of a mapping \( T \).

**Theorem 2.1.** Let \( X \) be a Banach space, let \( 0 \in C \subset X \) be nonempty, closed, convex and bounded. Let \( T : C \to C \) be a \( \alpha = (\alpha_1, \ldots, \alpha_n) \)-nonexpansive mapping such that

\[
\|T^i(\mu x) - T^i(\mu y)\| \leq \|T^i(\lambda x) - T^i(\lambda y)\| \quad \text{for any } x, y \in C, \quad 0 \leq \mu \leq \lambda, \quad i \in \{1, \ldots, n\}.
\]

Then \( d(T) = 0 \).

**Proof.** Fix \( k \geq 1 \). Define \( S_k := \left(1 - \frac{1}{k}\right)T \). Obviously, \( S_kx = \frac{1}{k} \cdot 0 + \left(1 - \frac{1}{k}\right)Tx \in C \). Then

\[
\|S_kx - S_ky\| = \left(1 - \frac{1}{k}\right)\|Tx - Ty\|.
\]

Next, we have:

\[
S_k^2x = S_k \left(\left(1 - \frac{1}{k}\right)Tx\right) = \left(1 - \frac{1}{k}\right)T \left(\left(1 - \frac{1}{k}\right)Tx\right);
\]

therefore, by assumptions:

\[
\|S_k^2x - S_k^2y\| = \left\|\left(1 - \frac{1}{k}\right)T \left(\left(1 - \frac{1}{k}\right)Tx - \left(1 - \frac{1}{k}\right)Ty\right)\right\| = \left(1 - \frac{1}{k}\right)\left\|T \left(\left(1 - \frac{1}{k}\right)Tx - \left(1 - \frac{1}{k}\right)Ty\right)\right\| \leq \left(1 - \frac{1}{k}\right)\|T^2x - T^2y\|.
\]
Similarly, for $i \geq 3$ we have:

$$S^i_k x = T_k^{-1} \left( \left( 1 - \frac{1}{k} \right) T x \right) = \left( 1 - \frac{1}{k} \right) T \left( \left( 1 - \frac{1}{k} \right) T \left( \ldots \left( 1 - \frac{1}{k} \right) T x \right) \ldots \right);$$

therefore:

$$\| S^i_k x - S^j_k y \| = \left( 1 - \frac{1}{k} \right) T \left( \left( 1 - \frac{1}{k} \right) T \left( \ldots \left( 1 - \frac{1}{k} \right) T x \right) \ldots \right) - \left( 1 - \frac{1}{k} \right) T \left( \ldots \left( 1 - \frac{1}{k} \right) T y \right) \ldots \right) \right),

\| S^i_k x - S^j_k y \| \leq \left( 1 - \frac{1}{k} \right) \left( 1 - \frac{1}{k} \right) \ldots \left( 1 - \frac{1}{k} \right) \| x - y \|.

By assumptions, $\sum_{i=1}^{n} \alpha_i \| T^i x - T^j x \| \leq \| x - y \|$ for some $\alpha = (\alpha_1, \ldots, \alpha_n)$ satisfying $0 \leq \alpha_i \leq 1$ and $\sum_{i=1}^{n} \alpha_i = 1$. Therefore:

$$\min \{ \| S^i_k x - S^j_k y \|, 1 \leq j \leq n \} \leq \sum_{i=1}^{n} \alpha_i \| S^i_k x - S^j_k y \| \leq \left( 1 - \frac{1}{k} \right) \| x - y \|.$$

By Theorem 1.2, $S_k$ has the unique fixed point. Denote this fixed point by $x_k$. We get:

$$\| x_k - T x_k \| = \| S_k x_k - T x_k \| = \left( 1 - \frac{1}{k} \right) T x_k - T x_k \| = \frac{1}{k} \| T x_k \| \to 0, \ k \to \infty,$$

this completes the proof.

For $k \geq 3$, there exists a mapping $T$ which does not satisfy the assumptions of Theorem 1.1; however, for $k \geq 2$, it satisfies the assumptions of Theorem 2.1. This will be illustrated by the following example:

**Example 2.2.** Fix $k \geq 2$. Let $\tau: [-1,1] \to [-1,1]$ be a non-decreasing function, having the Lipschitz constant $k(\tau) = k$, concave on $[-1,0]$, convex on $[0,1]$ and such that $\tau(0) = 0$. Now define $T: B_1 \ni x = (x_1, x_2, \ldots) \to Tx := (\tau(x_2), \frac{k}{k^2 - 1} x_3, x_4, x_5, \ldots) \in B_1$.

We will show that the assumptions of Theorem 1.1 are not satisfied for any multi-index $\alpha$ of length $n$. Notice, that:
\[ \|Tx - Ty\| = |\tau(x_2) - \tau(y_2)| + \frac{k}{k^2 - 1}|x_3 - y_3| + \sum_{i=4}^{\infty}|x_i - y_i| \]
\leq k|x_2 - y_2| + \frac{k}{k^2 - 1}|x_3 - y_3| + \sum_{i=4}^{\infty}|x_i - y_i|.

\[ \|T^2 x - T^2 y\| = |\tau\left(\frac{k}{k^2 - 1}x_3\right) - \tau\left(\frac{k}{k^2 - 1}y_3\right)| + \frac{k}{k^2 - 1}|x_4 - y_4| + \sum_{i=5}^{\infty}|x_i - y_i| \]
\leq \frac{k}{k^2 - 1}|x_3 - y_3| + \frac{k}{k^2 - 1}|x_4 - y_4| + \sum_{i=5}^{\infty}|x_i - y_i|.

Therefore, \( k(T) = k \) and \( k(T^i) = \frac{k^2}{k^2 - 1} > 1, \ i \geq 2. \)

It is easy to see that for \( k \geq 3 \), the assumptions of Theorem 1.1 are not satisfied for any multi-index \( \alpha \) of length \( n \). Indeed, we would need to have \( \alpha_1 \geq \frac{1}{2^{1-n}} \geq \frac{1}{2} = \frac{\sqrt{2}}{2} > \frac{1}{2} \)
and thus for any such \( \alpha = (\alpha_1, \ldots, \alpha_n) \), the mapping \( T \) would not be \( \alpha \)-nonexpansive.

We will now show that \( T \) satisfies the assumptions of Theorem 2.1; therefore, \( d(T) = 0. \)

It is enough to take \( n = 2 \). It is easy to check that
\[ \frac{1}{k}\|Tx - Ty\| + \frac{k-1}{k}\|T^2 x - T^2 y\| \leq \|x - y\|; \]
therefore, \( T \) is \( \left(\frac{1}{k}, \frac{k-1}{k}\right) \)-nonexpansive. We only have to show that
\[ \|T(\mu x) - T(\mu y)\| \leq \|T(\lambda x) - T(\lambda y)\| \quad \text{for any } x, y \in B_\delta, \ 0 \leq \mu \leq \lambda. \]

If \( |\tau(\mu x_2) - \tau(\mu y_2)| \leq |\tau(\lambda x_2) - \tau(\lambda y_2)| \), then obviously,
\[ \|T(\mu x) - T(\mu y)\| = |\tau(\mu x_2) - \tau(\mu y_2)| + \frac{k}{k^2 - 1}|\mu x_3 - \frac{k}{k^2 - 1}\mu y_3| + \sum_{i=4}^{\infty}|\mu x_i - \mu y_i| \]
\leq |\tau(\lambda x_2) - \tau(\lambda y_2)| + \frac{k}{k^2 - 1}|\lambda x_3 - \frac{k}{k^2 - 1}\lambda y_3| + \sum_{i=4}^{\infty}|\lambda x_i - \lambda y_i| \]
\[ = \|T(\lambda x) - T(\lambda y)\|. \]

It is enough to prove that \( |\tau(\mu v) - \tau(\mu w)| \leq |\tau(\lambda v) - \tau(\lambda w)| \) for \( v \neq w \) and \( 0 < \mu < \lambda. \)

Firstly, assume that \( v, w > 0 \). Without the loss of generality, we can assume that \( w < v \). Therefore, \( 0 < \mu w < \mu v, \lambda w < \lambda v. \)
Assume that $0 < \mu w < \mu v \leq \lambda w < \lambda v$. Choose $a \in (\lambda w, \lambda v]$ such that $a - \lambda w = \mu v - \mu w$. Of course, such an $a$ exists since $\lambda (v - w) \geq u (v - w)$. Therefore, $\mu v = \frac{a - \mu v}{a - \mu w} \mu w + \frac{\lambda w - \mu w}{a - \mu w} a$. Due to the convexity of $\tau$ on $[0,1]$

$$
\tau(\mu v) = \tau \left( \frac{a - \mu v}{a - \mu w} \mu w + \frac{\mu v - \mu w}{a - \mu w} a \right) \\
\leq \frac{a - \mu v}{a - \mu w} \tau(\mu v) + \frac{\mu v - \mu w}{a - \mu w} \tau(a)
$$

$$
\tau(\lambda w) = \tau \left( \frac{a - \lambda w}{a - \mu w} \mu w + \frac{\lambda w - \mu w}{a - \mu w} a \right) \\
\leq \frac{a - \lambda w}{a - \mu w} \tau(\mu v) + \frac{\lambda w - \mu w}{a - \mu w} \tau(a)
$$

$$
= \frac{\mu v - \mu w}{a - \mu w} \tau(\mu w) + \frac{a - \mu v}{a - \mu w} \tau(a)
$$

Adding the above estimates side-by-side and taking into consideration the fact that $\tau$ is non-decreasing, we get:

$$
\tau(\mu w) + \tau(\lambda w) \leq \left( \frac{a - \mu v}{a - \mu w} + \frac{\mu v - \mu w}{a - \mu w} \right) \tau(\mu w) + \left( \frac{\mu v - \mu w}{a - \mu w} + \frac{a - \mu v}{a - \mu w} \right) \tau(a)
$$

$$
= \tau(\mu w) + \tau(a) \leq \tau(\mu w) + \tau(\lambda w),
$$

this implies that $|\tau(\mu w) - \tau(\mu v)| \leq |\tau(\lambda v) - \tau(\lambda w)|$.

On the other hand, if $0 < \mu w < \lambda w \leq \mu v < \lambda v$, then let us choose $a \in (\mu v, \lambda v]$ such that $a - \mu v = \lambda w - \mu w$. Of course, such an $a$ exists since $(\lambda - \mu)v \geq (\lambda - \mu)w$. Then $\lambda w = \frac{a - \lambda w}{a - \mu w} \mu w + \frac{\lambda w - \mu w}{a - \mu w} a$ and $\mu v = \frac{a - \mu v}{a - \mu w} \mu w + \frac{\mu v - \mu w}{a - \mu w} a$. Due to the convexity of $\tau$ on $[0,1]$, we have:

$$
\tau(\lambda w) = \tau \left( \frac{a - \lambda w}{a - \mu w} \mu w + \frac{\lambda w - \mu w}{a - \mu w} a \right) \\
\leq \frac{a - \lambda w}{a - \mu w} \tau(\mu w) + \frac{\lambda w - \mu w}{a - \mu w} \tau(a)
$$
\[
\tau(\mu v) = \tau \left( \frac{a - \mu v}{a - \mu w} \frac{\mu v - \mu w}{a - \mu w} a \right)
\]
\[
\leq \frac{a - \mu v}{a - \mu w} \tau(\mu w) + \frac{\mu v - \mu w}{a - \mu w} \tau(a)
\]
\[
= \frac{\lambda w - \mu w}{a - \mu w} \tau(\mu y) + \frac{a - \lambda w}{a - \mu w} \tau(a)
\]

Again, adding the above estimations side-by-side, we get:
\[
\tau(\lambda w) + \tau(\mu v) \leq \left( \frac{a - \lambda w}{a - \mu w} + \frac{\lambda w - \mu w}{a - \mu w} \right) \tau(\mu w) + \left( \frac{\lambda w - \mu w}{a - \mu w} + \frac{a - \lambda w}{a - \mu w} \right) \tau(a)
\]
\[
= \tau(\mu w) + \tau(\lambda w),
\]

this leads to \(|\tau(\mu v) - \tau(\mu w)| \leq |\tau(\lambda v) - \tau(\lambda w)|\).

Similarly, it is easy to check that the estimation \(|\tau(\mu v) - \tau(\mu w)| \leq |\tau(\lambda v) - \tau(\lambda w)|\)
remains true for \(v, w < 0\) and for other cases. This shows that \(T\) satisfies the assumptions of Theorem 2.1.

A set satisfying \(\lambda x_0 + (1 - \lambda)y \in C\) for all \(y \in C\), \(\lambda \in [0,1]\) we call star-like set \(C\) with respect to \(x_0\).

**Theorem 2.2.** Let \(X\) be a Banach space, \(x_0 \in X\), \(N \in \mathbb{N}\), let \(C \subset X\) be a bounded, star-like set with respect to \(x_0\). Let \(T : C \rightarrow C\) be such that

1. \(\min \{\|T^j x - T^j y\|, 1 \leq j \leq N\} \leq \|x - y\|\) for all \(x, y \in C\),

2. there exists \(0 \leq b_0 \leq 1\) such that for all \(0 \leq b \leq b_0\), \(1 \leq j \leq N - 1\), \(x, y \in C\)
\[
\|T(T_b^j x) - T(T_b^j y)\| \leq (1 + b)\|T^{j+1} x - T^{j+1} y\|,
\]
where \(T_b x = (1-b)Tx + bx_0\). Then \(d(T) = 0\).

**Proof.** Fix arbitrary \(x, y \in C\) and take \(j \in \{1,\ldots, N\}\) such that \(\|T^j x - T^j y\| \leq \|x - y\|\).

Let us note that:
\[
\|T_b^j x - T_b^j y\| = \|T_b(T_b^{j-1} x) - T_b(T_b^{j-1} y)\|
\]
\[
= \|(1-b)T(T_b^{j-1} x) + bx_0 - (1-b)T(T_b^{j-1} y) - bx_0\|
\]
\[
= (1-b)\|T(T_b^{j-1} x) - T(T_b^{j-1} y)\|
\]
\[
\leq (1-b)(1+b)\|T^j x - T^j y\|
\]
\[
\leq (1-b^2)\|x - y\|
\]
Therefore, for any $x, y \in C$ there exists $j \in \{1, \ldots, N\}$ such that $\|T_b'x - T_b'y\| \leq (1 - b^2) \|x - y\|$. Theorem 1.2 ensures, that $T_b$ has the unique fixed point.

Now, fix an arbitrary $\varepsilon > 0$ and choose $0 \leq b \leq b_0$ such that $\|T_b z - Tz\| = \|(1 - b)Tz + bx_0 - Tz\| = b \|x_0 - Tz\| \leq \varepsilon$ for any $z \in C$.

Let $z_b \in C$ be such that $T_b z_b = z_b$.

Therefore, $\|z_b - Tz_b\| \leq \|z_b - T_b z_b\| + \|T_b z_b - Tz_b\| \leq 0 + \varepsilon = \varepsilon$, this proves that $d(T) = 0$.

Let us illustrate the possible application of Theorem 2.2.

**Example 2.3.** Let $T$ be the same as in Example 2.2. Then $T$ satisfies Theorem 2.2 (we have already shown that $T$ does not satisfy Theorem 1.1 for $k \geq 3$).

Indeed, let us calculate

\[
T(T_b x) = \left(\tau \left(1 - b\right) - \frac{k}{k^2 - 1} x_3\right), \frac{k}{k^2 - 1}(1 - b)x_4, (1 - b)x_5, (1 - b)x_6, \ldots
\]

and

\[
T^2 x = \left(\tau \left(-\frac{k}{k^2 - 1} x_3\right), \frac{k}{k^2 - 1} x_4, x_5, x_6, \ldots\right)
\]

We have:

\[
\|T(T_b x) - T(T_b y)\| = \left|\tau \left(1 - b\right) - \frac{k}{k^2 - 1} x_3\right| - \tau \left(-\frac{k}{k^2 - 1} y_3\right)
\]

\[
+ \left|\frac{k}{k^2 - 1}(1 - b)x_4 - \frac{k}{k^2 - 1}(1 - b)y_4\right| + \left|(1 - b)x_5 - (1 - b)y_5\right| + \ldots
\]

\[
\leq \left|\tau \left(-\frac{k}{k^2 - 1} x_3\right) - \tau \left(-\frac{k}{k^2 - 1} y_3\right)\right| + \left|\frac{k}{k^2 - 1} x_4 - \frac{k}{k^2 - 1} y_4\right| + \left|x_5 - y_5\right| + \ldots
\]

\[
= \|T^2 x - T^2 y\| \leq (1 + b) \|T^2 x - T^2 y\|
\]

We have already taken into account the fact, that $|\tau(\mu s) - \tau(\mu t)| \leq |\tau(\lambda s) - \tau(\lambda t)|$ for any $0 \leq \mu \leq \lambda, s, t \in [-1, 1]$. We proved this fact in Example 2.2.

The estimate $\min \left\{\|T x - T y\|, \|T^2 x - T^2 y\|\right\} \leq \frac{1}{k} \|T x - T y\| + \frac{k - 1}{k} \|T^2 x - T^2 y\| \leq \|x - y\|$ shows that $T$ satisfies Theorem 2.2.
References


