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PIOTR KOT*

# PEAK SET ON THE UNIT DISC

# ZBIÓR SZCZYTOWY DLA DYSKU JEDNOSTKOWEGO

A b s t r a c t

Abstract: We show that any compact subset $K$ in the boundary of the unit disc $\mathbb{D}$ with a zero measure is a peak set for $A(\mathbb{D})$.

*Keywords:*

S t r e s z c z e n i e

Pokażemy, że dowolny podzbiór zwarty $K$ miary zero w brzegu dysku jednostkowego jest zbiorem szczytowym dla $A(\mathbb{D})$.

*Słowa kluczowe:*

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* Piotr Kot (pkot@pk.edu.pl), Institute of Mathematics, Faculty of Physic, Mathematics and Computer Sciences, Cracow University of Technology.

## 1. Schwarz integral

The goal of this paper is to consider some properties of one-dimensional holomorphic functions in the unit disc. We focus our attention on such boundary properties of these functions which imply their uniqueness. In this aspect Luzin-Privalov theorem [4–6] seems to be crucial. This theorem refers to a meromorphic function $f(z)$ of the complex variable $z$ in a simply-connected domain $D$ with rectifiable boundary $\Gamma$. If $f(z)$ takes angular boundary values zero on a set $E \subset \Gamma$ of positive Lebesgue measure on $\Gamma$, then $f(z) = 0$ in D. There is no function meromorphic in $D$ that has infinite angular boundary values on a set $E \subset \Gamma$ of positive measure.

We are going to construct some examples of a holomorphic non-constant function $f$ for a given $E$ set of measure zero with $f = 1$ on $E$.

It will turn out that this $E$ set is a peak set for a proper algebra of holomorphic functions.

We say that a compact set $K$ is a peak set for $A(\mathbb{D})$ if there exists $f \in A(\mathbb{D})$ such that $|f| < 1$ on $\bar{\mathbb{D}} \setminus K$ and $f = 1$ on $K$. Stensönes Henriksen has proved [2] that every strictly pseudoconvex domain with $C^\infty$ boundary in $C^d$ has a peak set with a Hausdorff dimension $2d - 1$.

In this paper we give an alternative, even stronger construction for the unit disc. In the context of the Luzin-Privalov theorem we give the optimal construction for algebra $A(\mathbb{D})$.

Main tool in our construction is the Schwarz kernel.

Let us consider a natural measure $\sigma$ on boundary of the unit circle $\partial\mathbb{D}$. For a given $u$ which satisfies a Hölder condition we use Schwarz integral (see [7, 8]):

$$Su(z) := \frac{1}{2\pi i} \int_{\partial\mathbb{D}} u(t) \frac{t+z}{t-z} \frac{dt}{t}.$$

We can easily observe that $Su \in O(\mathbb{D})$.

Then the Schwarz integral formula $Su$ defining an analytic function, the boundary values of whose real part coincide with $u$. Additionally, the real part of $Su$ is a continuous harmonic function on $\bar{\mathbb{D}}$ (see [1, The Basic Lemma].

There exists a harmonic function $v$ on $\mathbb{D}$ so that $Su = u + iv$.

However when applying the above integral formula, a very important and more difficult problem arises concerning the existence and the expression of the boundary values of the imaginary part $v$ and of the complete function $Su$ by the given boundary values of the real part $u$. Still, in some cases we have complete information about $v$.

If a given function $u$ satisfies a Hölder condition, then the corresponding values of imaginary part $v$ on $\partial\mathbb{D}$ are expressed by the Hilbert formula (see [3, 1, pp. 45-49]):

$$v(\phi) = -\frac{1}{2\pi} \int_0^{2\pi} u(t) \cot\left(\frac{t-\varphi}{2}\right) dt.$$

The above formula is a singular integral and exists in the Cauchy principal-value sense.

Moreover, if $u$ satisfies a Hölder condition then the values of $v$ exist on all $\phi \in \partial\mathbb{D}$ and satisfy the same Hölder condition as $u$. Now we can recover $Su$ using $v$ in the following way:

$$Su(z) := \frac{1}{2\pi} \int_{\partial \mathsf{D}} v(t) \frac{t+z}{t-z} \frac{dt}{t} + c_1.$$

But now the imaginary part of $Su$ is continuous on $\bar{\mathsf{D}}$, so $Su \in A(\mathsf{D})$ if $u$ satisfies a Hölder condition.

## 2. Peak sets

**Lemma 1.** *Assume that $K$, $D$ are distinct compact sets in $\partial\mathsf{D}$. Then there exists a function $u \in C^\infty(\partial\mathsf{D})$ so that $u = 0$ on $D$, $u = 1$ on $K$ and $0 \le u \le 1$ on $\partial\mathsf{D}$.*

**Proof.** There exist open arcs $I_i : \{e^{2\pi it} : a_i < t < b_i\}$ such that $K \subset \bigcup_{i=1}^n I_i$ and $\overline{I}_i \cap D = \varnothing$. In fact we can assume that $\overline{I}_i \cap \overline{I}_j = \varnothing$ for $i \neq j$. Now there exist functions $u_i : \partial\mathsf{D} \to [0,1] \in C^\infty(\partial\mathsf{D})$ so that $u_i = 1$ on $I_i$, and $u_i = 0$ on $D$ but with distinct supports. It is enough to define $u = \sum_{k=1}^n u_k$.

**Theorem 2.** *Let $K$ be a compact subset of $\partial\mathsf{D}$ measure zero ($\sigma(K) = 0$). There exists a function $f \in A(\mathsf{D})$ such that* $|f| < 1$ on $\bar{\mathsf{D}} \setminus K$ and $f = 1$ on $K$.

**Proof.** Let us choose $\varepsilon > 0$ and define

$$D_\varepsilon := \{z \in \partial\mathsf{D} : \inf_{w \in K} |z - w| \ge \varepsilon\}$$

There exists $u_\varepsilon \in C^\infty(\partial\mathsf{D})$ such that $0 \le u_\varepsilon \le 1$, $u_\varepsilon(z) = 0$ if $z \in D_\varepsilon$ and $u_\varepsilon(z) = 1$ if $z \in K$. In particular $Su_\varepsilon \in A(\mathsf{D})$ and $0 \le \Re Su_\varepsilon \le 1$.

Let us choose $z \in \bar{\mathsf{D}} \setminus K$ and define $\delta(z,\varepsilon) := \inf_{w \in \partial\mathsf{D} \setminus D_\varepsilon} |z - w|$. We can estimate

$$|Su_\varepsilon(z)| \le \left| \frac{1}{2\pi} \int_{\partial\mathsf{D} \setminus D_\varepsilon} \frac{t+z}{t-z} \frac{dt}{t} \right| \le \frac{\sigma(\partial\mathsf{D} \setminus D_\varepsilon)}{2\pi} \max_{t \in U(\varepsilon)} \left| \frac{t+z}{t-z} \right| \le \frac{\sigma(\partial\mathsf{D} \setminus D_\varepsilon)}{\delta(z,\varepsilon)}.$$

Let us consider the following compact set:

$$T_n : \{z \in \bar{\mathsf{D}} : \inf_{w \in K} |z - w| \ge 2^{-n} + 2^{-2n}\}$$

There exists $\varepsilon_n \in (0, 2^{-2n})$ such that $\sigma(\partial\mathsf{D} \setminus D_{\varepsilon_n}) < 2^{-2n}$. Now let $g_n := Su_{\varepsilon_n} \in A(\mathsf{D})$.

Obviously $\Re g_n = 1$ on $K$ and $0 \le \Re g_n \le 1$.

Moreover if $z \in T_n$ then

$$|g_n(z)| \le \frac{\sigma(\partial\mathsf{D} \setminus D_{\varepsilon_n})}{\delta(z,\varepsilon_n)} \le \frac{2^{-2n}}{2^{-n} + 2^{-2n} - 2^{-2n}} = 2^{-n}.$$

Now we are able to define $g := 1 + \sum_{n \in N} g_n$.

Since $\bigcup_{n\in N} T_n = \bar{\mathsf{D}} \setminus K$ we can observe that $g \in O(\mathsf{D}) \cap C(\bar{\mathsf{D}} \setminus K)$. As $0 \le \Re g_n \le 1$ and $\Re g_n = 1$ on $K$ we have $\lim_{z\to w} \Re g_n(z) = \infty$ for $w \in K$.

Now we choose $f := \exp\left(-\frac{1}{g}\right)$. Obviously $f \in O(\mathsf{D}) \cap C(\bar{\mathsf{D}} \setminus K)$.

Since $\Re \frac{1}{g} = \frac{\Re \bar{g}}{|g|^2} = \frac{\Re g}{|g|^2} > 0$ on $\bar{\Omega} \setminus K$ we may easily observe that $0 < |f| < 1$ on $\bar{\Omega} \setminus K$. Additionally due to $\lim_{z\to w} \frac{1}{|g(z)|} = 0$ for $w \in K$ we have $f = 1$ on $K$ and $f \in A(\Omega)$.

**Example 3.** *There exists $K \subset \partial\mathsf{D}$, a compact set with Hausdorff dimension equal one which is also a peak set for $A(\mathsf{D})$.*

Let us consider a sequence of closed distinct intervals $I_n := [2^{-2n-1}, 2^{-2n}]$. There exists Cantor set $C_n \subset I_n$ with Hausdorf dimension equal $\frac{n}{n+1}$. Now we define a compact set

$$K := \{1\} \cup \bigcup_{n=1}^{\infty} \{e^{2\pi i t} : t \in C_n\}$$

in $\partial\mathsf{D}$ with Hausdorff dimension one and due to Theorem 2 we conclude that $K$ is a peak set for $A(\mathsf{D})$.

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