PIOTR KOT*

PEAK SET ON THE UNIT DISC

Abstract

Abstract: We show that any compact subset $K$ in the boundary of the unit disc $D$ with a zero measure is a peak set for $A(D)$.

Keywords:

ZBIÓR SZCZYTOWY DLA Dysku Jednostkowego

Streszczenie

Pokażemy, że dowolny podzbiór zwarty $K$ miary zero w brzegu dysku jednostkowego jest zbiorem szczytowym dla $A(D)$.

Słowa kluczowe:

DOI: 10.4467/2353737XCT.16.141.5752

* Piotr Kot (pkot@pk.edu.pl), Institute of Mathematics, Faculty of Physic, Mathematics and Computer Sciences, Cracow University of Technology.
1. Schwarz integral

The goal of this paper is to consider some properties of one-dimensional holomorphic functions in the unit disc. We focus our attention on such boundary properties of these functions which imply their uniqueness. In this aspect Luzin-Privalov theorem [4‒6] seems to be crucial. This theorem refers to a meromorphic function \( f(z) \) of the complex variable \( z \) in a simply-connected domain \( D \) with rectifiable boundary \( \Gamma \). If \( f(z) \) takes angular boundary values zero on a set \( E \subset \Gamma \) of positive Lebesgue measure on \( \Gamma \), then \( f(z) = 0 \) in \( D \). There is no function meromorphic in \( D \) that has infinite angular boundary values on a set \( E \subset \Gamma \) of positive measure.

We are going to construct some examples of a holomorphic non-constant function \( f \) for a given \( E \) set of measure zero with \( f = 1 \) on \( E \).

It will turn out that this \( E \) set is a peak set for a proper algebra of holomorphic functions.

We say that a compact set \( K \) is a peak set for \( A(D) \) if there exists \( f \in A(D) \) such that \(|f| < 1 \) on \( \overline{D} \setminus K \) and \( f = 1 \) on \( K \). Stensönes Henriksen has proved [2] that every strictly pseudoconvex domain with \( C^\infty \) boundary in \( \mathbb{C}^d \) has a peak set with a Hausdorff dimension \( 2d - 1 \).

In this paper we give an alternative, even stronger construction for the unit disc. In the context of the Luzin-Privalov theorem we give the optimal construction for algebra \( A(D) \).

Main tool in our construction is the Schwarz kernel.

Let us consider a natural measure \( \sigma \) on boundary of the unit circle \( \partial D \). For a given \( u \) which satisfies a Hölder condition we use Schwarz integral (see [7, 8]):

\[
Su(z) := \frac{1}{2\pi i} \int_{\partial D} \frac{u(t) \, dt}{t-z}.
\]

We can easily observe that \( Su \in O(D) \).

Then the Schwarz integral formula \( Su \) defining an analytic function, the boundary values of whose real part coincide with \( u \). Additionally, the real part of \( Su \) is a continuous harmonic function on \( D \) (see [1, The Basic Lemma]).

There exists a harmonic function \( v \) on \( D \) so that \( Su = u + iv \).

However when applying the above integral formula, a very important and more difficult problem arises concerning the existence and the expression of the boundary values of the imaginary part \( v \) and of the complete function \( Su \) by the given boundary values of the real part \( u \). Still, in some cases we have complete information about \( v \).

If a given function \( u \) satisfies a Hölder condition, then the corresponding values of imaginary part \( v \) on \( \partial D \) are expressed by the Hilbert formula (see [3, 1, pp. 45-49]):

\[
v(\phi) = -\frac{1}{2\pi} \int_{0}^{2\pi} u(t) \cot \left( \frac{t-\phi}{2} \right) dt.
\]

The above formula is a singular integral and exists in the Cauchy principal-value sense.

Moreover, if \( u \) satisfies a Hölder condition then the values of \( v \) exist on all \( \phi \in \partial D \) and satisfy the same Hölder condition as \( u \). Now we can recover \( Su \) using \( v \) in the following way:
\[ Su(z) = \frac{1}{2\pi} \int_{\partial D} v(t) \frac{t + z}{t - z} \, dt + c_1. \]

But now the imaginary part of \( Su \) is continuous on \( \overline{D} \), so \( Su \in A(D) \) if \( u \) satisfies a Hölder condition.

### 2. Peak sets

**Lemma 1.** Assume that \( K, D \) are distinct compact sets in \( \partial D \). Then there exists a function \( u \in C^\infty(\partial D) \) such that \( u = 0 \) on \( D \), \( u = 1 \) on \( K \) and \( 0 \leq u \leq 1 \) on \( \partial D \).

**Proof.** There exist open arcs \( I_i : \{ e^{2\pi it} : a_i < t < b_i \} \) such that \( K \subset \bigcup_{i=1}^n I_i \) and \( \overline{I}_i \cap D = \emptyset \). In fact we can assume that \( \overline{I}_i \cap \overline{I}_j = \emptyset \) for \( i \neq j \). Now there exist functions \( u_i : \partial D \to [0,1] \in C^\infty(\partial D) \) so that \( u_i = 1 \) on \( I_i \) and \( u_i = 0 \) on \( D \) but with distinct supports. It is enough to define \( u = \sum_{k=1}^n u_k \).

**Theorem 2.** Let \( K \) be a compact subset of \( \partial D \) measure zero (\( \sigma(K) = 0 \)). There exists a function \( f \in A(D) \) such that \( f < 1 \) on \( \partial D \backslash K \) and \( f = 1 \) on \( K \).

**Proof.** Let us choose \( \varepsilon > 0 \) and define
\[
D_\varepsilon := \{ z \in \partial D : \inf_{w \in K} |z - w| \geq \varepsilon \}
\]

There exists \( u_\varepsilon \in C^\infty(\partial D) \) such that \( 0 \leq u_\varepsilon \leq 1 \), \( u_\varepsilon(z) = 0 \) if \( z \in D_\varepsilon \) and \( u_\varepsilon(z) = 1 \) if \( z \in K \). In particular \( Su_\varepsilon \in A(D) \) and \( 0 \leq \Re Su_\varepsilon \leq 1 \).

Let us choose \( z \in \overline{D} \backslash K \) and define \( \delta(z, \varepsilon) := \inf_{w \in \partial D \backslash D_\varepsilon} |z - w| \). We can estimate
\[
|Su_\varepsilon(z)| \leq \frac{1}{2\pi} \left\{ \int_{\partial D \setminus D_\varepsilon} \frac{t + z}{t - z} \, dt \right\} \leq \frac{\sigma(\partial D \setminus D_\varepsilon)}{2\pi} \max_{t \in U(\varepsilon)} \frac{t + z}{t - z} \leq \frac{\sigma(\partial D \setminus D_\varepsilon)}{\delta(z, \varepsilon)}.
\]

Let us consider the following compact set:
\[
T_n := \{ z \in \overline{D} : \inf_{w \in K} |z - w| \geq 2^{-n} + 2^{-2n} \}
\]

There exists \( \varepsilon_n \in (0, 2^{-2n}) \) such that \( \sigma(\partial D \setminus D_{\varepsilon_n}) < 2^{-2n} \). Now let \( g_n := Su_{\varepsilon_n} \in A(D) \).

Obviously \( \Re g_n = 1 \) on \( K \) and \( 0 \leq \Re g_n \leq 1 \).

Moreover if \( z \in T_n \) then
\[
|g_n(z)| \leq \frac{\sigma(\partial D \setminus D_{\varepsilon_n})}{\delta(z, \varepsilon_n)} \leq \frac{2^{-2n}}{2^{-n} + 2^{-2n} - 2^{-2n}} = 2^{-n}.
\]

Now we are able to define \( g := 1 + \sum_{n \in \mathbb{N}} g_n \).
Since $\bigcup_{n \in N} T_n = \overline{D} \setminus K$ we can observe that $g \in O(D) \cap C(\overline{D} \setminus K)$. As $0 \leq \Re g_n \leq 1$ and $\Re g_n = 1$ on $K$ we have $\lim_{z \to w} \Re g_n(z) = \infty$ for $w \in K$.

Now we choose $f := \exp\left(-\frac{1}{g}\right)$. Obviously $f \in O(D) \cap C(\overline{D} \setminus K)$.

Since $\Re g = \frac{\Re g}{|g|^2} = \frac{\Re g}{|g|^2} > 0$ on $\overline{\Omega} \setminus K$ we may easily observe that $0 < |f| < 1$ on $\overline{\Omega} \setminus K$.

Additionally due to $\lim_{z \to w} \frac{1}{|g(z)|} = 0$ for $w \in K$ we have $f = 1$ on $K$ and $f \in A(\Omega)$.

**Example 3.** There exists $K \subset \partial D$, a compact set with Hausdorff dimension equal one which is also a peak set for $A(D)$.

Let us consider a sequence of closed distinct intervals $I_n := [2^{-2n-1}, 2^{-2n}]$. There exists Cantor set $C_n \subset I_n$ with Hausdorff dimension equal $\frac{n}{n+1}$. Now we define a compact set

$$K := \{1\} \cup \bigcup_{n=1}^{\infty} \{e^{2\pi it} : t \in C_n\}$$

in $\partial D$ with Hausdorff dimension one and due to Theorem 2 we conclude that $K$ is a peak set for $A(D)$.

**References**


