

Hugo ALBUQUERQUE, Josep Maria FONT,
Ramon JANSANA

NOTE OF THE FULL GENERALIZED MODELS OF THE EXTENSIONS OF A LOGIC

A b s t r a c t. In this short note we show that the full generalized models of any extension of a logic can be determined from the full generalized models of the base logic in a simple way. The result is a consequence of two central theorems of the theory of full generalized models of sentential logics. As applications we investigate when the full generalized models of an extension can also be full generalized models of the base logic, and we prove that each Suszko filter of a logic determines a Suszko filter of each of its extensions, also in a simple way.

We use the terminology and notations that are standard in abstract algebraic logic, as given for instance in [7].

We identify a *sentential logic* \mathcal{S} with a structural (i.e., substitution invariant) consequence relation $\vdash_{\mathcal{S}}$ on the set of formulas of some algebraic

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similarity type. For an algebra \mathbf{A} of the same type, the set (closure system) of all the \mathcal{S} -filters of \mathbf{A} is denoted by $\mathcal{F}i_{\mathcal{S}}\mathbf{A}$. For any $X \subseteq A$ we put $(\mathcal{F}i_{\mathcal{S}}\mathbf{A})^X := \{G \in \mathcal{F}i_{\mathcal{S}}\mathbf{A} : X \subseteq G\}$ and $Fg_{\mathcal{S}}^{\mathbf{A}}(X) := \bigcap (\mathcal{F}i_{\mathcal{S}}\mathbf{A})^X$. Thus, $Fg_{\mathcal{S}}^{\mathbf{A}}$ is the associated closure operator of \mathcal{S} -filter-generation.

For an algebra \mathbf{A} , its set (lattice) of congruences is denoted by $\text{Co}\mathbf{A}$, and given a class \mathbf{K} of algebras, $\text{Co}_{\mathbf{K}}\mathbf{A}$ is the set of congruences θ of \mathbf{A} such that $\mathbf{A}/\theta \in \mathbf{K}$; this set is ordered under set inclusion.

The central tool in this note is the Tarski operator, a construction performed on generalized matrices, which is best understood in the context of the Leibniz operator, a simpler construction performed on logical matrices.

Given an algebra \mathbf{A} , a congruence $\theta \in \text{Co}\mathbf{A}$ is *compatible* with a set $F \subseteq A$ when for every $a, b \in A$, if $\langle a, b \rangle \in \theta$ and $a \in F$, then $b \in F$. The largest of all congruences of \mathbf{A} compatible with F always exists and is known as the *Leibniz congruence of F* ; it is denoted by $\Omega^{\mathbf{A}}F$. Then, θ is compatible with F if and only if $\theta \subseteq \Omega^{\mathbf{A}}F$. Notice that these notions depend only on the algebraic structure of \mathbf{A} . A matrix $\langle \mathbf{A}, F \rangle$ is *reduced* when $\Omega^{\mathbf{A}}F = \text{Id}_A$. The *Leibniz operator* is the map $F \mapsto \Omega^{\mathbf{A}}F$; its significance for the algebraic study of a logic \mathcal{S} appears when its source is restricted to the $F \in \mathcal{F}i_{\mathcal{S}}\mathbf{A}$. The properties of this restricted operator have been exploited in the last decades in several directions, originating a classification of logics called *the Leibniz hierarchy* [6, 7].

A *generalized matrix*, or *g-matrix* for short, is a pair $\langle \mathbf{A}, \mathcal{C} \rangle$, where \mathbf{A} is an algebra and \mathcal{C} is a closure system over A . The *Tarski congruence* of the generalized matrix is

$$\tilde{\Omega}\langle \mathbf{A}, \mathcal{C} \rangle = \tilde{\Omega}^{\mathbf{A}}\mathcal{C} := \bigcap_{F \in \mathcal{C}} \Omega^{\mathbf{A}}F. \quad (1)$$

A g-matrix is reduced when $\tilde{\Omega}^{\mathbf{A}}\mathcal{C} = \text{Id}_A$. The *Tarski operator* is the map $\mathcal{C} \mapsto \tilde{\Omega}^{\mathbf{A}}\mathcal{C}$. These notions have parallel properties to those of the Leibniz-related ones. Again, their significance for a logic \mathcal{S} resides in their properties for the g-matrices that are models of the logic in the following sense: A g-matrix $\langle \mathbf{A}, \mathcal{C} \rangle$ is a *generalized model* (*g-model* for short) of a logic \mathcal{S} when $\mathcal{C} \subseteq \mathcal{F}i_{\mathcal{S}}\mathbf{A}$.

A truly general definition of the notion of the *algebraic counterpart* of a logic \mathcal{S} is the class $\text{Alg}\mathcal{S}$ of the algebraic reducts of the reduced g-models of \mathcal{S} (see [7, Section 5.4] for alternative definitions and a discussion). This class turns out to coincide with other classes traditionally associated with a

logic, under natural restrictions: for instance, for protoalgebraic logics [4, 6] it coincides with the class $\text{Alg}^* \mathcal{S}$ of algebraic reducts of the reduced matrix models of \mathcal{S} , and for algebraizable logics [5] it coincides with their equivalent algebraic semantics.

Among the g-models of a logic there are some of special importance to abstract algebraic logic. The *basic full g-models* of \mathcal{S} are the g-models of the form $\langle \mathbf{A}, \mathcal{F}i_{\mathcal{S}} \mathbf{A} \rangle$, that is, the “largest” g-models on the corresponding algebra. In some sense, these are the closest “images” of the logic (viewed as the closure system of its theories) on the algebra \mathbf{A} . One of the lines of research of abstract algebraic logic is the investigation of the properties of a logic that are inherited by its basic full g-models; examples are finitariness, the presence of theorems, the deduction-detachment theorem, etc. (see [7, Section 5.2] for details). Moreover, those of these properties that are preserved under strict surjective homomorphisms between g-matrices also hold for the so-called *full generalized models* of a logic; these are the g-matrices $\langle \mathbf{A}, \mathcal{C} \rangle$ that are inverse images of basic full g-models of the logic by strict surjective homomorphisms, that is, such that $\mathcal{C} = h^{-1}(\mathcal{F}i_{\mathcal{S}} \mathbf{B})$ for some algebra \mathbf{B} and some surjective $h: \mathbf{A} \rightarrow \mathbf{B}$.

The class of all full g-models of a logic \mathcal{S} is denoted by $\text{FGMod} \mathcal{S}$. The set of full g-models of \mathcal{S} over a fixed algebra \mathbf{A} is denoted by $\mathcal{FGMod}_{\mathcal{S}} \mathbf{A}$. One can prove that this set is a complete lattice when ordered under the set inclusion relation between the corresponding closure systems, so that the infimum of a family $\{\langle \mathbf{A}, \mathcal{C}_i \rangle : i \in I\} \subseteq \mathcal{FGMod}_{\mathcal{S}} \mathbf{A}$ is the g-matrix $\langle \mathbf{A}, \bigcap_{i \in I} \mathcal{C}_i \rangle$; to abbreviate, we say that $\mathcal{FGMod}_{\mathcal{S}} \mathbf{A}$ is “closed under intersections”.

$\text{Alg} \mathcal{S}$ is also the class of algebraic reducts of the reduced full g-models of \mathcal{S} ; since these turn out to be basic, they are exactly the g-matrices of the form $\langle \mathbf{A}, \mathcal{F}i_{\mathcal{S}} \mathbf{A} \rangle$ with $\mathbf{A} \in \text{Alg} \mathcal{S}$.

The theory of full g-models was started in the first edition of [9]; further details can be found in [7, Chapter 5], [10] and [11].

In this note we will only need to know two of their properties. First, the following interesting characterization ([9, Theorem 2.14]; see also [7, Proposition 5.94]).

Proposition 1. *A generalized matrix $\langle \mathbf{A}, \mathcal{C} \rangle$ is a full g-model of a logic \mathcal{S} if and only if $\mathcal{C} = \{F \in \mathcal{F}i_{\mathcal{S}} \mathbf{A} : \hat{\Omega}^{\mathbf{A}} \mathcal{C} \subseteq \Omega^{\mathbf{A}} F\}$.*

Notice that, in general, from the definition (1) of the Tarski congruence

it follows that $\tilde{\Omega}^{\mathbf{A}}\mathcal{C} \subseteq \Omega^{\mathbf{A}}F$ for any closure system \mathcal{C} and any $F \in \mathcal{C}$; moreover, since full g-models are g-models, they satisfy that $\mathcal{C} \subseteq \mathcal{F}i_{\mathcal{S}}\mathbf{A}$. Thus, Proposition 1 tells us that a g-model of \mathcal{S} is full if and only if its closure system is as large as it can be, given its Tarski congruence. This gives some sense to the term “full”, and highlights that full g-models are uniquely determined by their own Tarski congruence; that is, the map $\langle \mathbf{A}, \mathcal{C} \rangle \mapsto \tilde{\Omega}^{\mathbf{A}}\mathcal{C}$ is one-to-one over full g-models of a given logic \mathcal{S} . But there is more. One of the central results of the theory is the so-called “Isomorphism Theorem” ([9, Theorem 2.30]; see also [7, Theorem 5.95]).

Proposition 2. *For any logic \mathcal{S} and any algebra \mathbf{A} , the Tarski operator induces a dual order isomorphism from $\mathcal{F}GMod_{\mathcal{S}}\mathbf{A}$ to $\text{CoAlg}_{\mathcal{S}}\mathbf{A}$. The dual isomorphism is given by the map $\langle \mathbf{A}, \mathcal{C} \rangle \mapsto \tilde{\Omega}^{\mathbf{A}}\mathcal{C}$, and its inverse is the map $\theta \mapsto \langle \mathbf{A}, \{F \in \mathcal{F}i_{\mathcal{S}}\mathbf{A} : \theta \subseteq \Omega^{\mathbf{A}}F\} \rangle$.*

One of the most remarkable aspects of this theorem is its universal validity (in contrast, for instance, to the fact that the Leibniz operator is an isomorphism over $\mathcal{F}i_{\mathcal{S}}\mathbf{A}$ only for a weakly algebraizable \mathcal{S}).

The determination of the full g-models of a logic, and the quest to characterize them in terms of relevant metalogical properties of the logic, has been an important direction of research in abstract algebraic logic. To mention just a couple of examples, the full g-models of the implication fragment of intuitionistic logic are characterized by the deduction-detachment theorem [7, Example 5.91], and those of Belnap-Dunn’s four valued logic are characterized by the ordinary Gentzen-style rules of conjunction, disjunction, weak contraposition and double negation, but also by conditions of a semantical flavour that mimic the definition of the logic [7, Example 5.92].

An issue that has not been investigated up to now is whether there is some relation between the full g-models of a logic and those of its extensions.

If $\mathcal{S}, \mathcal{S}'$ are logics over the same language, \mathcal{S}' is an *extension* of \mathcal{S} when $\vdash_{\mathcal{S}} \subseteq \vdash_{\mathcal{S}'}$ or, equivalently, when every theory of \mathcal{S}' is a theory of \mathcal{S} ; this fact is denoted by $\mathcal{S} \leq \mathcal{S}'$. This situation at the level of the formula algebra is reflected at the algebraic level in the fact that $\mathcal{S} \leq \mathcal{S}'$ if and only if $\mathcal{F}i_{\mathcal{S}'}\mathbf{A} \subseteq \mathcal{F}i_{\mathcal{S}}\mathbf{A}$ for every algebra \mathbf{A} .

Our purpose is to show that, surprisingly, a parallel though more complicated situation appears concerning the full g-models of the two logics. In this case, $\mathcal{F}GMod_{\mathcal{S}'}\mathbf{A}$ is not a subset of $\mathcal{F}GMod_{\mathcal{S}}\mathbf{A}$, but each of the members of the former is given by a subset of a member of the latter, of a

fixed form, namely the intersection of (the closure system of) this one with $\mathcal{F}i_{S'}\mathbf{A}$.

Theorem 3. *Let $\mathcal{S}, \mathcal{S}'$ be two logics such that $\mathcal{S} \leq \mathcal{S}'$, and let \mathbf{A} be any algebra. Then $\langle \mathbf{A}, \mathcal{C} \rangle$ is a full g-model of \mathcal{S}' if and only if there exists a full g-model $\langle \mathbf{A}, \mathcal{D} \rangle$ of \mathcal{S} such that $\mathcal{C} = \mathcal{D} \cap \mathcal{F}i_{S'}\mathbf{A}$.*

Moreover, the \mathcal{D} can be so chosen that $\tilde{\Omega}^{\mathbf{A}}\mathcal{D} = \tilde{\Omega}^{\mathbf{A}}\mathcal{C}$.

In symbols, $\text{FGMod}\mathcal{S}' = \{ \langle \mathbf{A}, \mathcal{D} \cap \mathcal{F}i_{S'}\mathbf{A} \rangle : \langle \mathbf{A}, \mathcal{D} \rangle \in \text{FGMod}\mathcal{S} \}$.

Proof. Assume first that $\langle \mathbf{A}, \mathcal{C} \rangle \in \text{FGMod}\mathcal{S}'$. By Proposition 1, this means that $\mathcal{C} = \{ F \in \mathcal{F}i_{S'}\mathbf{A} : \tilde{\Omega}^{\mathbf{A}}\mathcal{C} \subseteq \Omega^{\mathbf{A}}F \}$. Define

$$\mathcal{D} := \{ F \in \mathcal{F}i_{S'}\mathbf{A} : \tilde{\Omega}^{\mathbf{A}}\mathcal{C} \subseteq \Omega^{\mathbf{A}}F \}. \quad (2)$$

Clearly, $\mathcal{C} \subseteq \mathcal{D} \cap \mathcal{F}i_{S'}\mathbf{A}$, because $\mathcal{F}i_{S'}\mathbf{A} \subseteq \mathcal{F}i_{S}\mathbf{A}$. Conversely, if $F \in \mathcal{D} \cap \mathcal{F}i_{S'}\mathbf{A}$, it follows directly that $F \in \mathcal{C}$. Hence, $\mathcal{C} = \mathcal{D} \cap \mathcal{F}i_{S'}\mathbf{A}$ as wanted. Moreover, the definition (1) of the Tarski operator implies that it is anti-monotonic on the closure systems; in this case, since $\mathcal{C} \subseteq \mathcal{D}$, $\tilde{\Omega}^{\mathbf{A}}\mathcal{D} \subseteq \tilde{\Omega}^{\mathbf{A}}\mathcal{C}$. On the other hand, by (2), $\tilde{\Omega}^{\mathbf{A}}\mathcal{C} \subseteq \Omega^{\mathbf{A}}F$ for every $F \in \mathcal{D}$; using (1) to define $\tilde{\Omega}^{\mathbf{A}}\mathcal{D}$, this implies that $\tilde{\Omega}^{\mathbf{A}}\mathcal{C} \subseteq \tilde{\Omega}^{\mathbf{A}}\mathcal{D}$. Thus, $\tilde{\Omega}^{\mathbf{A}}\mathcal{D} = \tilde{\Omega}^{\mathbf{A}}\mathcal{C}$, as announced. Now (2) can be rewritten as $\mathcal{D} = \{ F \in \mathcal{F}i_{S}\mathbf{A} : \tilde{\Omega}^{\mathbf{A}}\mathcal{D} \subseteq \Omega^{\mathbf{A}}F \}$, and this tells us by Proposition 1 that $\langle \mathbf{A}, \mathcal{D} \rangle \in \text{FGMod}\mathcal{S}$, as wanted. This proves the implication from left to right and the subsequent observation.

In order to prove the converse, assume that $\langle \mathbf{A}, \mathcal{D} \rangle \in \text{FGMod}\mathcal{S}$ and take $\mathcal{C} := \mathcal{D} \cap \mathcal{F}i_{S'}\mathbf{A}$. We need only prove that $\langle \mathbf{A}, \mathcal{C} \rangle \in \text{FGMod}\mathcal{S}'$, and by Proposition 1 this amounts to showing that

$$\mathcal{C} = \{ F \in \mathcal{F}i_{S'}\mathbf{A} : \tilde{\Omega}^{\mathbf{A}}\mathcal{C} \subseteq \Omega^{\mathbf{A}}F \}. \quad (3)$$

If $F \in \mathcal{C}$, obviously $F \in \mathcal{F}i_{S'}\mathbf{A}$, and, as observed before as a general property, $\tilde{\Omega}^{\mathbf{A}}\mathcal{C} \subseteq \Omega^{\mathbf{A}}F$. Conversely, let $F \in \mathcal{F}i_{S'}\mathbf{A}$ be such that $\tilde{\Omega}^{\mathbf{A}}\mathcal{C} \subseteq \Omega^{\mathbf{A}}F$. Since $\mathcal{C} \subseteq \mathcal{D}$, by the anti-monotonicity of the Tarski operator, $\tilde{\Omega}^{\mathbf{A}}\mathcal{D} \subseteq \tilde{\Omega}^{\mathbf{A}}\mathcal{C}$, and thus $\tilde{\Omega}^{\mathbf{A}}\mathcal{D} \subseteq \Omega^{\mathbf{A}}F$. Since $\langle \mathbf{A}, \mathcal{D} \rangle$ is a full g-model of \mathcal{S} and $F \in \mathcal{F}i_{S'}\mathbf{A} \subseteq \mathcal{F}i_{S}\mathbf{A}$, it follows by Proposition 1 that $F \in \mathcal{D}$. So, $F \in \mathcal{C}$. This shows (3), and thus completes the proof that $\langle \mathbf{A}, \mathcal{C} \rangle \in \text{FGMod}\mathcal{S}'$. \square

This characterization is especially interesting when \mathcal{S}' is the extension of \mathcal{S} given by some concrete set of axioms or inference rules; in this case,

the full g-models of \mathcal{S}' are obtained by taking the full g-models of \mathcal{S} and keeping the \mathcal{S} -filters that are closed under those axioms or inference rules.

The formulation of Theorem 3 stresses that the \mathcal{D} in the statement need not be unique. Indeed, in the second part of the proof, given \mathcal{D} and having defined $\mathcal{C} := \mathcal{D} \cap \mathcal{F}i_{\mathcal{S}'}\mathbf{A}$, it is not possible in general to prove that $\tilde{\Omega}^{\mathbf{A}}\mathcal{D} = \tilde{\Omega}^{\mathbf{A}}\mathcal{C}$; but applying the construction in the first part of the proof to this \mathcal{C} one would find another \mathcal{D}' such that $\tilde{\Omega}^{\mathbf{A}}\mathcal{D}' = \tilde{\Omega}^{\mathbf{A}}\mathcal{C}$. A concrete, very simple example of this situation is obtained by considering $\mathcal{I}PC$, intuitionistic propositional logic, and $\mathcal{C}PC$, classical propositional logic. Certainly $\mathcal{I}PC \leq \mathcal{C}PC$, and since they are (weakly) algebraizable logics, we know how their full g-models look like [7, Theorem 6.117]: they have the form $\langle \mathbf{A}, (\mathcal{F}i_{\mathcal{S}}\mathbf{A})^F \rangle$ for an arbitrary $F \in \mathcal{F}i_{\mathcal{S}}\mathbf{A}$, for $\mathcal{S} \in \{\mathcal{I}PC, \mathcal{C}PC\}$. Now, take the 3-element Heyting algebra with universe $A := \{0, a, 1\}$ as \mathbf{A} . If $\mathcal{C} := \{A\}$, $\mathcal{D} := \{\{a, 1\}, A\}$ and $\mathcal{D}' := \{A\}$, then $\langle \mathbf{A}, \mathcal{C} \rangle$ is a full g-model of $\mathcal{C}PC$ and both $\langle \mathbf{A}, \mathcal{D} \rangle$ and $\langle \mathbf{A}, \mathcal{D}' \rangle$ are full g-models of $\mathcal{I}PC$. Since A is the only $\mathcal{C}PC$ -filter of \mathbf{A} , we have that $\mathcal{D} \cap \mathcal{F}i_{\mathcal{C}PC}\mathbf{A} = \mathcal{D}' \cap \mathcal{F}i_{\mathcal{C}PC}\mathbf{A} = \mathcal{C}$. Clearly, the second part of the proof gives $\mathcal{D} \mapsto \mathcal{C}$, and $\tilde{\Omega}^{\mathbf{A}}\mathcal{D}$ identifies only a and 1 , while $\tilde{\Omega}^{\mathbf{A}}\mathcal{C}$ is the total relation on A . On the other hand, it is easy to check that the first part of the proof gives $\mathcal{C} \mapsto \mathcal{D}'$ and that $\tilde{\Omega}^{\mathbf{A}}\mathcal{C} = \tilde{\Omega}^{\mathbf{A}}\mathcal{D}'$, as expected.

It is thus natural to wonder when is this correspondence unique. The answer is that this happens if and only if the two logics share the same algebraic counterpart.

Theorem 4. *Let $\mathcal{S}, \mathcal{S}'$ be two logics such that $\mathcal{S} \leq \mathcal{S}'$. The following conditions are equivalent.*

- (i) $\text{Alg}\mathcal{S} = \text{Alg}\mathcal{S}'$.
- (ii) *For every algebra \mathbf{A} , the set $\mathcal{F}G\text{Mod}_{\mathcal{S}}\mathbf{A}$ is order isomorphic to the set $\mathcal{F}G\text{Mod}_{\mathcal{S}'}\mathbf{A}$, the isomorphism is given by the map $\langle \mathbf{A}, \mathcal{D} \rangle \mapsto \langle \mathbf{A}, \mathcal{D} \cap \mathcal{F}i_{\mathcal{S}'}\mathbf{A} \rangle$, and it is such that $\tilde{\Omega}^{\mathbf{A}}\mathcal{D} = \tilde{\Omega}^{\mathbf{A}}(\mathcal{D} \cap \mathcal{F}i_{\mathcal{S}'}\mathbf{A})$.*

Proof. (i) \Rightarrow (ii): We apply the Isomorphism Theorem (Proposition 2) to each of the logics, first directly to \mathcal{S} and then in the reverse direction to \mathcal{S}' . We obtain two dual order isomorphisms:

$$\begin{array}{ll} \mathcal{F}G\text{Mod}_{\mathcal{S}}\mathbf{A} \cong^{\text{D}} \text{CoAlg}_{\mathcal{S}}\mathbf{A} & \text{CoAlg}_{\mathcal{S}'}\mathbf{A} \cong^{\text{D}} \mathcal{F}G\text{Mod}_{\mathcal{S}'}\mathbf{A} \\ \langle \mathbf{A}, \mathcal{D} \rangle \mapsto \tilde{\Omega}^{\mathbf{A}}\mathcal{D} & \theta \mapsto \langle \mathbf{A}, \{F \in \mathcal{F}i_{\mathcal{S}'}\mathbf{A} : \theta \subseteq \Omega^{\mathbf{A}}F\} \rangle \end{array}$$

The assumption in (i) implies that $\text{CoAlg}_{\mathcal{S}}\mathbf{A} = \text{CoAlg}_{\mathcal{S}'}\mathbf{A}$, and therefore we can compose the two dual order isomorphisms and obtain an (ordinary) order isomorphism from $\mathcal{FGMod}_{\mathcal{S}}\mathbf{A}$ to $\mathcal{FGMod}_{\mathcal{S}'}\mathbf{A}$, given by the map

$$\langle \mathbf{A}, \mathcal{D} \rangle \mapsto \langle \mathbf{A}, \{F \in \mathcal{Fis}'\mathbf{A} : \tilde{\Omega}^{\mathbf{A}}\mathcal{D} \subseteq \Omega^{\mathbf{A}}F\} \rangle.$$

Since $\langle \mathbf{A}, \mathcal{D} \rangle$ is a full g-model of \mathcal{S} , by Proposition 1 we know that $\mathcal{D} = \{F \in \mathcal{Fis}\mathbf{A} : \tilde{\Omega}^{\mathbf{A}}\mathcal{D} \subseteq \Omega^{\mathbf{A}}F\}$. Therefore, using that $\mathcal{Fis}'\mathbf{A} \subseteq \mathcal{Fis}\mathbf{A}$, we conclude that the closure system in the rightmost g-matrix is exactly $\mathcal{D} \cap \mathcal{Fis}'\mathbf{A}$, as wanted. Finally, Proposition 2 again, applied to $\mathcal{D} \cap \mathcal{Fis}'\mathbf{A}$, directly gives that $\tilde{\Omega}^{\mathbf{A}}\mathcal{D} = \tilde{\Omega}^{\mathbf{A}}(\mathcal{D} \cap \mathcal{Fis}'\mathbf{A})$.

(ii) \Rightarrow (i): From the assumption that $\mathcal{S} \leq \mathcal{S}'$ it follows that $\text{Alg}\mathcal{S}' \subseteq \text{Alg}\mathcal{S}$ and that $\mathcal{Fis}'\mathbf{A} \subseteq \mathcal{Fis}\mathbf{A}$ for every \mathbf{A} . Now let $\mathbf{A} \in \text{Alg}\mathcal{S}$. This means that $\langle \mathbf{A}, \mathcal{Fis}\mathbf{A} \rangle \in \mathcal{FGMod}_{\mathcal{S}}\mathbf{A}$ and $\tilde{\Omega}^{\mathbf{A}}(\mathcal{Fis}\mathbf{A}) = \text{Id}_{\mathbf{A}}$. Now, $\mathcal{Fis}\mathbf{A} \cap \mathcal{Fis}'\mathbf{A} = \mathcal{Fis}'\mathbf{A}$, so that, by the last point in the assumption in (ii), $\tilde{\Omega}^{\mathbf{A}}(\mathcal{Fis}'\mathbf{A}) = \tilde{\Omega}^{\mathbf{A}}(\mathcal{Fis}\mathbf{A}) = \text{Id}_{\mathbf{A}}$. But this implies that $\mathbf{A} \in \text{Alg}\mathcal{S}'$. \square

By inspection of the proof of Theorem 3 we can see that, under the assumption of (i) in Theorem 4, the first map defined in Theorem 3 is exactly the inverse of the one given in (ii) of Theorem 4.

It is interesting to notice that the situation described in this theorem cannot appear when the logic \mathcal{S} is truth-equational, for there can be no proper extension \mathcal{S}' of a truth-equational logic \mathcal{S} such that $\text{Alg}\mathcal{S} = \text{Alg}\mathcal{S}'$. The reason lies in the fact that the extensions of a truth-equational logic \mathcal{S} are truth-equational as well, and have the same set τ of defining equations as \mathcal{S} . Therefore, if \mathcal{S} is truth-equational, $\mathcal{S} \leq \mathcal{S}'$ and $\text{Alg}\mathcal{S} = \text{Alg}\mathcal{S}'$, then the two logics are the τ -assertional logic of the same class of algebras, and therefore \mathcal{S} and \mathcal{S}' are equal. Proposition 7 below describes another assumption that leads to the same conclusion.

One general situation where the above results apply is that of the pairs of an arbitrary logic \mathcal{S} and its *strong version* \mathcal{S}^+ , a notion introduced and studied in [1, 3]. It may be defined, among several ways, as the logic determined by the class of all matrices whose filter is the smallest \mathcal{S} -filter on the corresponding algebra. The quoted publications show that the logic \mathcal{S}^+ is an extension of \mathcal{S} with a privileged status among all its extensions, and Theorem 3 tells us how to find its full g-models from those of \mathcal{S} . Moreover, it often happens, although this is not a general fact, that $\text{Alg}\mathcal{S} = \text{Alg}\mathcal{S}^+$; in these cases, the stronger Theorem 4 applies and the correspondence

between the full g-models of \mathcal{S} and those of \mathcal{S}^+ is optimal. Several notable, large classes of examples of this situation are reviewed in [3], for instance those where \mathcal{S} is the logic that preserves degrees of truth with respect to an arbitrary variety of commutative integral residuated lattices, and \mathcal{S}^+ is the logic that preserves truth with respect to the same variety. Notice that in those cases the logic \mathcal{S} is not truth-equational while the logic \mathcal{S}^+ is so.

Now we draw some consequences of the main results. The first one is mainly of a methodological interest.

Proposition 5. *Let \mathcal{S} and \mathcal{S}' be two logics such that $\mathcal{S} \leq \mathcal{S}'$ and let \mathbf{A} be any algebra. The following conditions are equivalent.*

- (i) $\langle \mathbf{A}, \mathcal{F}i_{\mathcal{S}'} \mathbf{A} \rangle$ is a full g-model of \mathcal{S} .
- (ii) Every full g-model of \mathcal{S}' over \mathbf{A} is a full g-model of \mathcal{S} .

Proof. (i) \Rightarrow (ii): By Theorem 3, every full g-model of \mathcal{S}' on \mathbf{A} has the form $\langle \mathbf{A}, \mathcal{D} \cap \mathcal{F}i_{\mathcal{S}'} \mathbf{A} \rangle$ for some full g-model $\langle \mathbf{A}, \mathcal{D} \rangle$ of \mathcal{S} . By (i) and the fact that $\mathcal{FG}Mod_{\mathcal{S}} \mathbf{A}$ is closed under intersections we obtain that $\langle \mathbf{A}, \mathcal{D} \cap \mathcal{F}i_{\mathcal{S}'} \mathbf{A} \rangle$ is also a full g-model of \mathcal{S} . Trivially, (ii) \Rightarrow (i), because $\langle \mathbf{A}, \mathcal{F}i_{\mathcal{S}'} \mathbf{A} \rangle$ is always a full g-model of \mathcal{S}' . \square

Observe that a proof of the (weaker) fact that condition (i) holds for all algebras if and only if condition (ii) holds for all algebras can be easily obtained from the most elementary definitions and general properties of the notion of a full g-model; moreover, it is also easy to see that these global properties hold if and only if they hold in the algebras in the class $\mathbf{Alg} \mathcal{S}$. Thus, the main interest of Proposition 5 is its limitation to a single, arbitrary algebra.

As a consequence of Proposition 5, if \mathcal{S}' is an extension of \mathcal{S} and every basic full g-model of \mathcal{S}' is a full g-model of \mathcal{S} , then the same holds for all full g-models of \mathcal{S}' . One may wonder how common this property is. The following partial result provides a very large class of examples.

Proposition 6. *Let \mathcal{S} and \mathcal{S}' be two logics such that $\mathcal{S} \leq \mathcal{S}'$. Assume moreover that \mathcal{S} is truth-equational and that \mathcal{S}' is an axiomatic extension of \mathcal{S} . Then all full g-models of \mathcal{S}' are full g-models of \mathcal{S} .*

Proof. By Proposition 5 we need just check that for an arbitrary \mathbf{A} , the g-matrix $\langle \mathbf{A}, \mathcal{F}i_{\mathcal{S}'} \mathbf{A} \rangle$ is a full g-model of \mathcal{S} . Take $F_{\circ} := \bigcap \mathcal{F}i_{\mathcal{S}'} \mathbf{A} \in$

$\mathcal{F}i_{\mathcal{S}'}\mathbf{A} \subseteq \mathcal{F}i_{\mathcal{S}}\mathbf{A}$. We claim that $\mathcal{F}i_{\mathcal{S}'}\mathbf{A} = (\mathcal{F}i_{\mathcal{S}}\mathbf{A})^{F_{\circ}}$. By construction we have that $\mathcal{F}i_{\mathcal{S}'}\mathbf{A} \subseteq (\mathcal{F}i_{\mathcal{S}}\mathbf{A})^{F_{\circ}}$. Conversely, assume that $F \in \mathcal{F}i_{\mathcal{S}}\mathbf{A}$ with $F_{\circ} \subseteq F$. Since F_{\circ} is an \mathcal{S}' -filter, $h(\varphi) \in F_{\circ} \subseteq F$ for all formulas φ in the set of axioms defining \mathcal{S}' out of \mathcal{S} and every evaluation h into \mathbf{A} . Therefore, F is also an \mathcal{S}' -filter, i.e., $F \in \mathcal{F}i_{\mathcal{S}'}\mathbf{A}$. This proves the claim. Now, the assumption that \mathcal{S} is truth-equational implies [7, Theorem 6.104] that any g-matrix of the form $\langle \mathbf{A}, (\mathcal{F}i_{\mathcal{S}}\mathbf{A})^{F_{\circ}} \rangle$ with $F_{\circ} \in \mathcal{F}i_{\mathcal{S}}\mathbf{A}$ is a full g-model of \mathcal{S} . Therefore, $\langle \mathbf{A}, \mathcal{F}i_{\mathcal{S}'}\mathbf{A} \rangle$ is a full g-model of \mathcal{S} , as wanted. \square

Notice that if \mathcal{S} is truth-equational, then all its extensions are truth-equational as well. Therefore, this result encompasses a large class of pairs of logics, one being an axiomatic extension of the other, and the two being truth-equational; in particular, this includes the cases where the two are algebraizable, such as that of \mathcal{IPC} and \mathcal{CPC} used before as a counterexample of another point, and many others.

The property is however not universal. In fact, we can show that in the domain of application of Theorem 4 (which, as already commented, is very large) the property never holds (save for the trivial case $\mathcal{S} = \mathcal{S}'$).

Proposition 7. *Let \mathcal{S} and \mathcal{S}' be two logics such that $\mathcal{S} \leq \mathcal{S}'$ and $\text{Alg}\mathcal{S} = \text{Alg}\mathcal{S}'$. If the full g-models of \mathcal{S}' are full g-models of \mathcal{S} , then $\mathcal{S} = \mathcal{S}'$.*

Proof. Let \mathbf{A} be any algebra. We know that $\langle \mathbf{A}, \mathcal{F}i_{\mathcal{S}}\mathbf{A} \rangle$ is a full g-model of \mathcal{S} and that $\langle \mathbf{A}, \mathcal{F}i_{\mathcal{S}'}\mathbf{A} \rangle$ is a full g-model of \mathcal{S}' . By the assumption, $\langle \mathbf{A}, \mathcal{F}i_{\mathcal{S}'}\mathbf{A} \rangle$ is also a full g-model of \mathcal{S} . By Theorem 4, $\tilde{\Omega}^{\mathbf{A}}(\mathcal{F}i_{\mathcal{S}}\mathbf{A}) = \tilde{\Omega}^{\mathbf{A}}(\mathcal{F}i_{\mathcal{S}}\mathbf{A} \cap \mathcal{F}i_{\mathcal{S}'}\mathbf{A})$. Since $\mathcal{F}i_{\mathcal{S}}\mathbf{A} \cap \mathcal{F}i_{\mathcal{S}'}\mathbf{A} = \mathcal{F}i_{\mathcal{S}'}\mathbf{A}$, we conclude that $\tilde{\Omega}^{\mathbf{A}}(\mathcal{F}i_{\mathcal{S}}\mathbf{A}) = \tilde{\Omega}^{\mathbf{A}}(\mathcal{F}i_{\mathcal{S}'}\mathbf{A})$. Thus, we have two full g-models of \mathcal{S} on the same algebra with the same Tarski congruence. By the Isomorphism Theorem (Proposition 2), they must be equal, i.e., $\mathcal{F}i_{\mathcal{S}}\mathbf{A} = \mathcal{F}i_{\mathcal{S}'}\mathbf{A}$. Since \mathbf{A} is arbitrary, this implies that $\mathcal{S} = \mathcal{S}'$. \square

There are other examples where the conclusion of Proposition 6 does not hold and are not covered by Proposition 7. To mention just one, consider Positive Modal Logic \mathcal{PML} as analyzed in [2, Section 4] and its strong version \mathcal{PML}^+ , which turns out to be the extension of \mathcal{PML} with the Rule of Necessitation $x \vdash \Box x$. It is proved that $\text{Alg}\mathcal{PML} \neq \text{Alg}\mathcal{PML}^+$. The logic \mathcal{PML} is fully selfextensional, which means that all its full g-models have the property of congruence (see [7, Chapter 7]). Now, if all the full g-models of \mathcal{PML}^+ were full g-models of \mathcal{PML} , then they would satisfy

the property of congruence, and as a consequence \mathcal{PML}^+ would be fully selfextensional as well. But this is not the case; in fact, \mathcal{PML}^+ is not even selfextensional, a much weaker property.

The second application regards the *Suszko filters* of a logic. This notion has been introduced in [1, 2], where it is studied in depth, as an instance of more general notions concerning compatibility operators, and for its relevance in several characterizations of some classes of the Leibniz hierarchy. Here, all we need to know about it is the following characterization [2, Theorem 5.13] in terms of the full g-models of the logic of a particular kind.

Proposition 8. *Let \mathcal{S} be a logic, and $F \in \mathcal{Fis}\mathbf{A}$. Then F is a Suszko \mathcal{S} -filter if and only if the g-matrix $\langle \mathbf{A}, (\mathcal{Fis}\mathbf{A})^F \rangle$ is a full g-model of \mathcal{S} .*

Notice that not all full g-models of a logic need to be given by a closure system that is a principal filter in the lattice $\mathcal{Fis}\mathbf{A}$; actually, this property characterizes protoalgebraic logics [7, Theorem 6.39]. In general, a logic will have as many Suszko filters as full g-models of this kind. A logic is truth-equational if and only if all its filters are Suszko filters, i.e., if and only if all its filters define a full g-model in the above way [7, Theorem 6.104].

The characterization of the full g-models of the extensions of a logic in Theorem 3 allows us to obtain the next result, relating the Suszko filters of a logic with those of its extensions.

Theorem 9. *Let $\mathcal{S}, \mathcal{S}'$ be two logics such that $\mathcal{S} \leq \mathcal{S}'$ and let \mathbf{A} be any algebra. For each $F \in \mathcal{Fis}\mathbf{A}$ define $G := Fg_{\mathcal{S}'}^{\mathbf{A}}(F)$. If F is a Suszko \mathcal{S} -filter, then G is a Suszko \mathcal{S}' -filter. If moreover $\text{Alg}\mathcal{S} = \text{Alg}\mathcal{S}'$, then $\tilde{\Omega}^{\mathbf{A}}((\mathcal{Fis}\mathbf{A})^F) = \tilde{\Omega}^{\mathbf{A}}((\mathcal{Fis}\mathbf{A})^G)$.*

Proof. By Proposition 8 the g-matrix $\langle \mathbf{A}, (\mathcal{Fis}\mathbf{A})^F \rangle$ is a full g-model of \mathcal{S} . If we put $\mathcal{C} := \mathcal{Fis}_{\mathcal{S}'}\mathbf{A} \cap (\mathcal{Fis}\mathbf{A})^F$, then by Theorem 3 the g-matrix $\langle \mathbf{A}, \mathcal{C} \rangle$ is a full g-model of \mathcal{S}' . But, since $\mathcal{Fis}_{\mathcal{S}'}\mathbf{A} \subseteq \mathcal{Fis}\mathbf{A}$, it is clear that $\mathcal{C} = \{H \in \mathcal{Fis}_{\mathcal{S}'}\mathbf{A} : F \subseteq H\}$, so that $G = Fg_{\mathcal{S}'}^{\mathbf{A}}(F) = \bigcap \mathcal{C}$, and then actually $\mathcal{C} = \{H \in \mathcal{Fis}_{\mathcal{S}'}\mathbf{A} : G \subseteq H\} = (\mathcal{Fis}_{\mathcal{S}'}\mathbf{A})^G$. Now, we have that the g-matrix $\langle \mathbf{A}, (\mathcal{Fis}_{\mathcal{S}'}\mathbf{A})^G \rangle$ is a full g-model of \mathcal{S}' . By Proposition 8 applied to \mathcal{S}' , we conclude that G is a Suszko \mathcal{S}' -filter. Finally, if we add the assumption that $\text{Alg}\mathcal{S} = \text{Alg}\mathcal{S}'$, then we can apply Theorem 4, obtaining that $\tilde{\Omega}^{\mathbf{A}}((\mathcal{Fis}\mathbf{A})^F) = \tilde{\Omega}^{\mathbf{A}}((\mathcal{Fis}\mathbf{A})^F \cap \mathcal{Fis}_{\mathcal{S}'}\mathbf{A})$. Therefore, $\tilde{\Omega}^{\mathbf{A}}((\mathcal{Fis}\mathbf{A})^F) = \tilde{\Omega}^{\mathbf{A}}((\mathcal{Fis}_{\mathcal{S}'}\mathbf{A})^G)$. \square

Recall that for protoalgebraic logics the notion of a Suszko filter coincides with the older one of a *Leibniz filter* [8]; thus, in this case we would obtain the analogue of Theorem 9 for Leibniz filters of a protoalgebraic logic.

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Hugo Albuquerque and Ramon Jansana
Departament de Filosofia,
Universitat de Barcelona (UB)
Montalegre 7, E-08001 Barcelona, Spain
hugo.albuquerque@ua.pt jansana@ub.edu

Josep Maria Font
Departament de Matemàtiques i Informàtica,
Universitat de Barcelona (UB)
Gran Via de les Corts Catalanes 585, E-08007 Barcelona, Spain
jmfont@ub.edu