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COMPLEMENTARY PAIR OF QUASI-ANTIORDERS

A b s t r a c t. The aims of the present paper are to introduction and investigate of notions of complementary pairs of quasi-antiorders and half-space quasi-antiorder on a given set. For a pair α and β of quasi-antiorders on a given set A we say that they are complementary pair if $\alpha \cup \beta = \neq_A$ and $\alpha \cap \beta = \emptyset$. In that case, α (and β) is called half-space on A. Assertion, if α is a half-space quasi-antiorder on A, then the induced anti-order θ on $A/(\alpha \cup \alpha^{-1})$ is a half-space too, is the main result of this paper.

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1. Introduction

This paper is a continuation of the corresponding author's recent papers [4], [9], [10], and [11]. Our setting is Bishop's constructive mathematics ([1], [2], [6], [12]).

The concept of a relational system was introduced by A.I.Maltsev ([5]) and developed by many mathematicians (see, for example [3]). We will restrict our consideration to relational systems with only one binary relation. Hence, by a relational system we will take a pair $\boldsymbol{A} = (A, R)$, where $(A, =, \neq)$ is a set with apartness and $R \subseteq A \times A$, i. e., R is a binary relation on A. Relational systems play an important role both in mathematics and in applications since every formal description of a real system can be done by means of relations. For these considerations we often ask about a certain factorization of a relational system $\boldsymbol{A} = (A, R)$ because it enables us to introduce the method of abstraction on $\boldsymbol{A}$. Hence, if q is a coequality on A, we ask about a 'factor relation' R/q on the factor set A/q such that the factor system $(A/q, R/q)$ shares some of 'good' properties of $\boldsymbol{A}$.

In this paper, we are mostly interested in relational systems $\boldsymbol{A} = (A, R)$ where R is *consistent*, i.e. $(\forall x, y \in A)((x, y) \in R \Longrightarrow x \neq y)$ and *cotransitive*, i.e. $(a, c) \in R$ imply $(\forall b \in A)((a, b) \in R \vee (b, c) \in R)$. In that case, $\boldsymbol{A}$ is called a *consistent and cotransitive system* or a *quasi-antiorder system*. Our intention is to study the situation on $\boldsymbol{A}$ such that the system $(A/q, R/q)$ is also consistent and cotransitive.

Let us note that a similar task for anti-ordered sets was already studied in [4], [9]-[11]. According to [9] and [10], if $(S, =, \neq, \cdot, \alpha)$ is an anti-ordered semigroup and σ a quasi-antiorder on S, then the relation q on S, defined by $q = \sigma \cup \sigma^{-1}$, is an anticongruence on S and the set S/q is an anti-ordered semigroup under anti-order θ defined by $(xq, yq) \in \theta \Longleftrightarrow (x, y) \in \sigma$.

2. Preliminaries

Let $(A, =, \neq)$ be a set in the sense of books [1], [2], [6] and [12], where " $\neq$ " is a binary relation on A which satisfies the following properties:

$$\neg(x \neq x),\ x \neq y \Longrightarrow y \neq x,\ x \neq z \Longrightarrow x \neq y \vee y \neq z,$$

$$x \neq y \wedge y = z \Longrightarrow x \neq z,$$

called *apartness* (A. Heyting). Let Y be a subset of A and $x \in A$. The subset Y of A is *strongly extensional* in A if and only if $y \in Y \Longrightarrow y \neq x \vee x \in Y$ ([1], [2]). We define ([7]-[11]) $x \bowtie Y$ by $(\forall y \in Y)(y \neq x)$ and $Y^C = \{x \in A : x \bowtie Y\}$. For a subset Y of A we say that it is a *detachable subset* of A if the following $x \in A \Longrightarrow x \in Y \vee x \bowtie Y$ holds ([12]).

Let $\alpha \subseteq A \times B$ and $\beta \subseteq B \times C$ be relations. The *filled product* ([7], [8]) of relations α and β is the relation

$$\beta * \alpha = \{(a, c) \in A \times C : (\forall\, b \in B)((a, b) \in \alpha \vee (b, c) \in \beta)\}.$$

It is easy to check that the filled product is associative. (See, for example, [8]) For $\beta = \alpha$ we put ${}^2\alpha = \alpha * \alpha$, and for given natural n, by induction, we define

$${}^{n+1}\alpha = {}^n\alpha * \alpha\,(= \alpha * {}^n\alpha), \;\; {}^1\alpha = \alpha.$$

Besides, for any relation $\alpha \subseteq X \times X$, we can construct the relation

$$c(\alpha) = \bigcap_{n \in N} {}^n\alpha.$$

It is clear that $c(\alpha) \subseteq \alpha$ and the following $c(\alpha) \subseteq c(\alpha) * c(\alpha)$ is valid. It is called *cotransitive internal fulfilment* of α. This notion was studied by the third author in his articles [7], [8] and [11]. If α is a consistent relation on set A, then $c(\alpha)$ is the maximal quasi-antiorder on A under α (see, for example, article [7] or Theorem 3 in [11]).

A relation $q \subseteq A \times A$ is a *coequality relation* on A if and only if holds:

$$q \subseteq \neq, \;\; q \subseteq q^{-1}, \;\; q \subseteq q * q.$$

If q is a coequality relation on set $(A, =, \neq)$, we can construct factor-set $(A/q, =_1, \neq_1)$ with

$$aq =_1 bq \Longleftrightarrow (a, b) \bowtie q, \;\; aq \neq_1 bq \Longleftrightarrow (a, b) \in q.$$

A relation α on A is *antiorder* ([9]-[11]) on A if and only if

$$\alpha \subseteq \neq, \;\; \alpha \subseteq \alpha * \alpha, \;\; \neq \subseteq \alpha \cup \alpha^{-1}.$$

Antiorder α is a *linear antiorder* if $\alpha \cap \alpha^{-1} = \emptyset$ holds. As in [9], a relation $\tau \subseteq A \times A$ is a *quasi-antiorder* on A if and only if

$$\tau \subseteq (\alpha \subseteq) \neq, \;\; \tau \subseteq \tau * \tau.$$

It is easy to check that (quasi-)antiorder is a strongly extensional subset of $A \times A$. Let us note that families $\Im(A)$ of all quasi-antiorders on set A is a completely lattice. Indeed, in the following lemma we give proof for this fact:

Lemma 0 *If* $\{\tau_k\}_{k\in J}$ *is a family of quasi-antiorders on a set* $(A, =, \neq)$*, then* $\cup_{k\in J}\tau_k$ *and* $c(\cap_{k\in J}\tau_k)$ *are quasi-antiorders in A. So, the family* $\Im(X)$ *is a completely lattice.*

Proof: (1) Let $\{\tau_k\}_{k\in J}$ be a family of quasi-antiorders on a set $(A, =, \neq)$ and let x, z be arbitrary elements of A such that $(x, z) \in \cup_{k\in J}\tau_k$. Then, there exists k in J such that $(x, z) \in \tau_k$. Hence, for every $y \in A$ we have $(x, y) \in \tau_k \vee (y, z) \in \tau_k$. So, $(x, y) \in \cup_{kJ}\tau_k \vee (y, z) \in \cup_{kJ}\tau_k$. At the other side, for every k in J holds $\tau_k \subseteq \neq$. From this we have $\cup_{k\in J}\tau_k \subseteq \neq$. So, we can put $\bigvee\{\tau_k : k \in J\} = \cup_{k\in J}\tau_k$.
(2) Let $R(\subseteq\neq)$ be a relation on a set $(A, =, \neq)$. Then for an inhabited family of quasi-antiorders under R there exists the biggest quasi-antiorder relation under R. That relation is exactly the relation $c(R)$. In fact:
By (1), there exists the biggest quasi-antiorder relation on A under R.
Let $\boldsymbol{Q}_R$ be the inhabited family of all quasi-antiorder relation on A under R. With (R) we denote the biggest quasi-antiorder relation $\cup\boldsymbol{Q}_R$ on X under R. At the other side, the fulfillment $c(R) = \cap_{n\in N}\,{}^nR$ of the relation R is a cotransitive relation on set A under R. Therefore, $c(R) \subseteq (R)$ holds. We need to show that $(R) \subseteq c(R)$. Let $\tau\,(\subseteq (R) = \cup\boldsymbol{Q}_R)$ be a quasi-antiorder relation in A under R. The first, we have $\tau \subseteq R = {}^1R$. Let $(x, z) \in \tau$. Then from $(\forall y \in X)((x, y) \in \tau \vee (y, z) \in \tau)$ we conclude that for every y in X holds $(x, y) \in R \vee (y, z) \in R$, i.e. holds $(x, z) \in R * R = {}^2R$. So, $\tau \subseteq {}^2R$. Now, we will suppose that ${}^nR$ and let $(x, z) \in \tau$. Then from $(\forall y \in X)((x, y) \in \tau \vee (y, z) \in \tau)$ implies that $(x, y) \in R \vee (y, z) \in {}^nR$ holds for every $y \in A$. Therefore, $(x, z) \in {}^{n+1}R$. So, we have $\tau \subseteq {}^{n+1}R$. Thus, by induction, we have $\tau \subseteq {}^nR$ for any natural n. Remember that τ is an arbitrary quasi-antiorder on A under R. Hence, we proved that $(R) = \cup\boldsymbol{Q}_R \subseteq c(R)$. If $\{\tau_k\}_{k\in J}$ is a family of quasi-antiorders on a set $(A, =, \neq)$, then $c(\cap_{k\in J}\tau_k)$ is a quasi-antiorder in A, and we can set $\bigwedge\{\tau_k : k \in J\} = c(\cap_{k\in J}\tau_k)$. □

3. Complementary pair of quasi-antiorders

A pair of quasi-antiorders $\alpha \subseteq A \times A$ and $\beta \subseteq A \times A$ is said to be a *complementary pair* of quasi-antiorders if $\alpha \cup \beta = \neq_A$ and $\alpha \cap \beta = \emptyset$ holds. In this case, for α we say that it is a *half-space* (of $\neq_A$). Clearly, the complement β is also a half-space. The simplest examples of half-spaces are: linear antiorders, the apartness $\neq_A$ and the empty relation on any set A. Complementary pair of quasi-antiorders are put into a pair of the form $\alpha \perp \beta \,(\Longleftrightarrow \beta \perp \alpha)$ and can be characterized in the lattice $(\Im(A), \cup, \wedge)$ of all quasi-antiorders on A as follows.

Theorem 1. *For any quasi-antiorders* $\alpha, \beta \in \Im(A)$ *the following are equivalent*:
(1) $\alpha \perp \beta$,
(2) $\alpha \cup \beta = \neq_A$ *and* $(\alpha \cup \gamma) \wedge (\beta \cup \gamma) = \gamma$ *for all* $\gamma \in \Im(A)$.

Proof. (1) $\Longrightarrow$ (2):

$$\gamma = \emptyset \cup \gamma = (\alpha \cap \beta) \cup \gamma = (\alpha \cup \gamma) \cap (\beta \cup \gamma) \supseteq (\alpha \cup \gamma) \wedge (\beta \cup \gamma) \supseteq \gamma.$$

(2) $\Longrightarrow$ (1): For $\gamma = \emptyset$, we have $\alpha \wedge \beta = (\alpha \cup \emptyset) \wedge (\beta \cup \emptyset) = \emptyset$. Suppose that $\alpha \cap \beta \neq \emptyset$, then there exists $(a, b) \in \alpha \cap \beta$ for some $a, b \in A$. Let us prove first that $\gamma = (\alpha \cup \beta) \setminus \{(a, b)\}$ is a quasi-antiorder on A. Let (u, w) be an arbitrary element of γ and let v be an element of A. Then $(u, w) \neq (a, b)$, and hence $u \neq a \vee w \neq b$. Thus, we have $(u \neq a \vee v \neq b) \vee (v \neq a \vee w \neq b)$. Hence, the implication $(u, w) \in \gamma \Longrightarrow (u, v) \in \gamma \vee (v, w) \in \gamma$ is valid. Second, since γ is a quasi-antiorder on A, we have $(\alpha \cup \gamma) \wedge (\beta \cup \gamma) = \gamma \subset \neq_A$. It is a contradiction, because we have $\alpha \cup \gamma = \neq_A$ and $\beta \cup \gamma = \neq_A$. Indeed, let (u, v) be an arbitrary element of the apartness $\neq_A$. Since α is a strongly extensional subset of $\neq_A$, we have that out of $(a, b) \in \alpha$ implies $(a, b) \neq (u, v)$ or $(u, v) \in \alpha$. Thus, $(u, v) \in \gamma$ or $(u, v) \in \alpha$. So, $\neq_A = \alpha \cup \gamma$. The proof of assertion $\neq_A = \beta \cup \gamma$ we get analogously. $\square$

Example. Let $\alpha = \{(c, a), (c, b), (d, a), (d, b), (d, c), (e, a), (e, b), (e, c)\}$ and $\beta = \{(a, b), (a, c), (a, d), (a, e), (b, a), (b, c), (b, d), (b, e), (c, d), (c, e), (d, e), (e, d)\}$ be relations on set $A = \{a, b, c, d, e\}$. Then α and β are quasi-antiorders on A such that $\alpha \cap \alpha^{-1} = \emptyset$, $\alpha \cup \alpha^{-1} \subset \neq_A$,

$$\beta \cap \beta^{-1} = \{(b, a), (a, b), (e, d), (d, e)\},$$

$\beta \cup \beta^{-1} = \neq_A$, $\alpha \cup \beta = \neq_A$ and $\alpha \cap \beta = \emptyset$. So, the pair (α, β) is a nontrivial complementary pair of quasi-antiorders on A.

Note. Let x, y, z be elements of A and let α be a half-space quasi-antiorder relation on A. Then, holds $(x, y) \in \alpha^C \cap \neq_A$ and $(y, z) \in \alpha^C \cap \neq_A$ implies $(x, z) \in \alpha^C$. Indeed, if (u, v) be an arbitrary element of α, then we have

$$\begin{aligned}(u, v) \in \alpha &\Longrightarrow (u, x) \in \alpha \vee (x, y) \in \alpha \vee (y, z) \in \alpha \vee (z, v) \in \alpha \\ &\Longrightarrow u \neq x \vee z \neq v \\ &\Longrightarrow (x, z) \neq (u, v) \in \alpha.\end{aligned}$$

For a half-space α the inverse relation α^{-1} is also a half-space, and if $\alpha \bot \beta$ for $\alpha, \beta \in \Im(A)$, then $\alpha^{-1} \bot \beta^{-1}$. If $B \subseteq A$ is a subset, then the restriction of a quasi-antiorder to B yields a quasi-antiorder on B and a similar statement holds for half-spaces, $\alpha \bot \beta$ implies that $\alpha \cap (B \times B) \bot \beta \cap (B \times B)$.

Theorem 2. *For a quasi-antiorder $\alpha, \beta \in \Im(A)$ the following assertion is valid*:
(1) *If α is a half-space then for any x, y of A holds*

$$x \neq y \Longrightarrow (x, y) \in \alpha \vee (x, y) \bowtie \alpha.$$

(2) *If α and β are complementary pair of quasi-antiorders on A, then $\beta = c(\alpha^C \cap \neq_A)$ holds, i.e. relation β is the maximal quasi-antiorder on A under the relation $\alpha^C \cap \neq_A$.*

Proof. (1) Let α is a half-space quasi-antiorder in A and let β be a quasi-antiorder in A such that $\alpha \bot \beta$, i.e. such that $\neq_A = \alpha \cup \beta$ and $\alpha \cap \beta = \emptyset$. Thus, if $x \neq y$, then $(x, y) \in \alpha$ or $(x, y) \in \beta$. In the second case, we have $\neg((x, y) \in \alpha)$. Hence, if (u, v) be an arbitrary element of α, then $(u, x) \in \alpha$ or $(x, y) \in \alpha$ or $(y, v) \in \alpha$. Therefore, we have $(x, y) \neq (u, v) \in \alpha$, in the second case. So, $(x, y) \bowtie \alpha$.
(2) Firstly, the relation $c(\alpha^C \cap \neq_A)$ is the maximal quasi-antiorder relation on A under set $\alpha^C \cap \neq_A$ such that $c(\alpha^C \cap \neq_A) \subseteq \beta$. Secondly, if (u, v) is an arbitrary element of β, then we have $u \neq v$ and by (1) of this lemma, $(u, v) \in \alpha$ or $(u, v) \bowtie \alpha$. Thus, by elementary property of operator c ([7]), we have $\beta = c(\beta) \subseteq c(\alpha^C \cap \neq_A)$. □

As corollary of above assertion we have that any half-space quasi-antiorder on set A is a detachable subset of $A \times A$.

Let α be a half-space quasi-antiorder in a set A. Then ([10]) the relation $q = \alpha \cup \alpha^{-1}$ is a coequality relation on A and the factor-set $A/(\alpha \cup \alpha^{-1})$ is ordered under induced anti-order θ, defined by $(aq, bq) \in \theta$ if and only if $(a, b) \in \alpha$. In the following theorem we show that induced anti-order θ is a half-space, too.

Theorem 3. *If α is a half-space quasi-antiorder on A, then the induced anti-order θ is a half-space on $A/(\alpha \cup \alpha^{-1})$ also.*

Proof: Put $q = \alpha \cup \alpha^{-1}$. If we take

$$B = \{a \in A : (\exists b \in A)((a, b) \in \alpha \vee (b, a) \in \alpha)\},$$

then $\alpha \cap (B \times B)$ is a half-space quasi-antiorder in B and there exists a complementary half-space β' on B of $\alpha \cap (B \times B)$ such that $\neq_B = (\alpha \cap (B \times B)) \cup \beta'$ and $(\alpha \cap (B \times B)) \cap \beta' = \emptyset$. Let us define θ' on $A/(\alpha \cup \alpha^{-1})$ by $(uq, vq) \in \theta'$ if and only if $(u, v) \in \beta'$. It is easy to check that θ' is a quasi-antiorder on $A/(\alpha \cup \alpha^{-1})$. Thus, for arbitrary element (aq, bq) of A/q, if holds $aq \neq_1 bq$, we have $(a, b) \in \alpha \cup \alpha^{-1}$. Hence, we conclude that $a, b \in B$ and $a \neq_B b$. So, by definition of complementary pair of half-space, we have $(a, b) \in \alpha \cap (B \times B)$ or $(a, b) \in \beta'$. It means $(aq, bq) \in \theta$ or $(aq, bq) \in \theta'$. The proof for $\theta \cap \theta' = \emptyset$ we obtain simply. □

References

[1] E. Bishop, Foundations of constructive analysis, McGraw-Hill, New York 1967.

[2] D. S. Bridges and F. Richman, Varieties of constructive mathematics, London Mathematical Society Lecture Notes 97, Cambridge University Press, Cambridge, 1987.

[3] I. Cajda, *Congruences in transitive relational systems*, Miscolc Math. Notes **5** (1) (2004), pp. 19–23.

[4] D. Jojić and D.A.Romano, *Quasi-antiorder relational systems*, Inter. J. Contem. Math. Sciences **3** (27) (2008), pp. 1307–1315.

[5] A.I. Maltsev, Algebraic systems, Springer - Verlag, Berlin, 1973.

[6] R. Mines, F. Richman and W. Ruitenburg, A course of constructive algebra, Springer, New York 1988.

[7] D.A. Romano, *On construction of maximal coequality relation and its applications*, Proceedings of 8th international conference on Logic and Computers Sciences "LIRA '97", Novi Sad, September 1-4, 1997, (Editors: R.Tošić and Z.Budimac), Institute of Mathematics, Novi Sad 1997, pp. 225–230.

[8] D.A. Romano, *Some relations and subsets of semigroup with apartness generated by the principal consistent subsets*, Univ. Beograd, Publ. Elektroteh. Fak. Ser. Math. **13** (2002), pp. 7–25.

[9] D.A. Romano, *A note on quasi-antiorder in semigroup*, Novi Sad J. Math. **37** (1) (2007), pp. 3–8.

[10] D.A. Romano, *On regular anticongruence in anti-ordered semigroups*, Publications de l'Institut Mathematique **81** (95) (2007), pp. 95–102.

[11] D.A. Romano, *An isomorphism theorem for anti-ordered sets*, Filomat **22** (1) (2008), pp. 145–160.

[12] A.S. Troelstra and D. van Dalen, Constructivism in mathematics, an Introduction, North-Holland, Amsterdam 1988.

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