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PERMANENCE AND POSITIVE BOUNDED SOLUTIONS OF KOLMOGOROV PREDATOR-PREY SYSTEM

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Abstract. Our main purpose is to present some criteria for the permanence and existence of a positive bounded solution of Kolmogorov predator--prey system. Under certain conditions, it is shown that the system is permanent and there exists a solution which is defined on the whole $\mathbb{R}$ and whose components are bounded from above and from below by positive constants.

1. Introduction. We consider the following Kolmogorov predator-prey system

$$\text{(1.1)} \qquad \begin{cases} \dot{u}_i &= u_i f_i(t, u_1, \dots, u_n, v_1, \dots, v_m), \ i = 1, \ \dots, \ n, \\ \dot{v}_j &= v_j h_j(t, u_1, \dots, u_n, v_1, \dots, v_m), \ j = 1, \ \dots, \ m, \end{cases}$$

where $f_i, h_j : \mathbb{R} \times \mathbb{R}_+^{n+m} \to \mathbb{R}$ are continuous, $u_i(t)$ denotes the quantity of the i^{th} prey at time t and $v_j(t)$ denotes the quantity of the j^{th} predator at time t.

A special case of (1.1) is the system of Lotka–Volterra type:

$$\text{(1.2)} \qquad \begin{cases} \dot{u}_i &= u_i \left[b_i(t) - \sum_{k=1}^n a_{ik}(t)u_k - \sum_{k=1}^m c_{ik}(t)v_k\right], \ i = 1, \dots, n, \\ \dot{v}_j &= v_j \left[r_j(t) + \sum_{k=1}^n d_{jk}(t)u_k - \sum_{k=1}^m e_{jk}(t)v_k\right], \ j = 1, \dots, m, \end{cases}$$

where $a_{ik}(t)$, $c_{ik}(t)$, $d_{jk}(t)$, $e_{jk}(t)$, $b_i(t)$, $r_j(t)$ are continuous and bounded on $\mathbb{R}$.

A fundamental ecological question associated with the study of multispecies population interactions is the long-term coexistence of the involved populations. Such questions also arise in many other situations (see [**3**]). Mathematically, this is equivalent to the so-called permanence of the populations. We

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recall that *system (1.1) is permanent if there exist positive constants* δ *and* Δ ($\delta < \Delta$) *such that any noncontinuable solution* $(u_1(.), \dots, u_n(.), v_1(.), \dots, v_m(.))$ *of (1.1) with* $(u_1(t_0), \dots, u_n(t_0), v_1(t_0), \dots, v_m(t_0)) \in \operatorname{int} \mathbb{R}_+^{n+m}$ *– the interior of* $\mathbb{R}_+^{n+m}$*, is defined on* $[t_0, +\infty)$ *and for* $i = 1, \dots, n,\ j = 1, \dots, m$ *the following inequalities are satisfied:*

$$\delta \leqslant \liminf_{t\to+\infty} u_i(t) \leqslant \limsup_{t\to+\infty} u_i(t) \leqslant \Delta, \quad \delta \leqslant \liminf_{t\to+\infty} v_j(t) \leqslant \limsup_{t\to+\infty} v_j(t) \leqslant \Delta.$$

The permanence, the existence and global attractivity of a positive periodic solution of system (1.1) and (1.2) in the periodic case have been studied by Wen and Wang (see [**6**]), as well as many other authors. Some results on sufficient conditions for the existence and global attractivity of a unique positive almost periodic solution of system (1.2) in the almost periodic case were mentioned in [**7**]. For the Kolmogorov competing system, the authors in [**5**] have obtained a sufficient condition for the permanence and the existence of a positive bounded solution. As a continuation of [**5–7**] and some recent results, in this paper we study the permanence and the existence of a positive bounded solution of the Kolmogorov predator-prey system under certain conditions. The paper is organized as follows: Section 2 contains preliminaries, in which we present the relevant results on the permanence and asymptotic behaviour of solutions of a single-species model. In Section 3, we prove our main result on the permanence and existence of a positive bounded solution of system (1.1). In the last section, we study the permanence, existence and global attractivity of a unique positive almost periodic solution of Lotka–Volterra system (1.2).

2. Preliminaries. Consider the following equation

$$\dot{x} = xg(t, x), \tag{2.1}$$

where $g : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ is continuous. Let $\mathbb{R}_+ =: [0, +\infty)$. We assume that:

(G_1) The function $g(., 0)$ is bounded and $\lim_{x\to 0} \{\sup_{t\in\mathbb{R}} |g(t, x) - g(t, 0)|\} = 0$,

(G_2) There exists $\lambda > 0$ such that $\liminf_{t\to+\infty} \int_t^{t+\lambda} g(s, 0)ds > 0$,

(G_3) There exist a positive number ω and a function $a : \mathbb{R} \to \mathbb{R}_+$, which is bounded and locally integrable with $\liminf_{t\to+\infty} \int_t^{t+\omega} a(s)ds > 0$ such that $D_x^+ g(t, x)) \leqslant -a(t)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}_+$, where D_x^+ is the upper right derivative with respect to x.

Let $\mathcal{B}_+ = \{b : \mathbb{R} \to \mathbb{R}$ is continuous and $0 < \inf_{t\in\mathbb{R}} b(t) \leqslant \sup_{t\in\mathbb{R}} b(t) < +\infty\}$.

Lemma 2.1. *If $g(t,x)$ is nonincreasing in x, then for each initial value $x(t_0) = x_0 \in \mathbb{R}_+$, equation (2.1) has a unique solution $x(t)$ for $t \geqslant t_0$.*

Proof. By the way of contradiction we assume that there exists $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}_+$ such that there are two distinct solutions $x_1(t)$ and $x_2(t)$ on $[t_0, t_1]$ $(t_1 > t_0)$ of (2.1) with $x_1(t_0) = x_2(t_0) = x_0$. Without loss of generality, we may assume that $x_1(t) > x_2(t)$ for $t \in (t_0, t_1]$. There are two possible cases:

+) If $x_0 > 0$ then $[\ln x_1(t) - \ln x_2(t)]' = g(t, x_1(t)) - g(t, x_2(t)) \leqslant 0$ for all $t \in [t_0, t_1]$. Hence, $0 < \ln x_1(t_1) - \ln x_2(t_1) \leqslant \ln x_1(t_0) - \ln x_2(t_0) = 0$. This is a contradiction.

+) If $x_0 = 0$ then $x_1(t) > 0$ for all $t \in (t_0, t_1]$. Hence, $\dot{x}_1(t) = x_1(t) g(t, x_1(t)) \leqslant \gamma x_1(t)$ for $t \in [t_0, t_1]$ and for some $\gamma > 0$. By Gronwall's inequality, $x_1(t) = 0$ for all $t \in [t_0, t_1]$. This is a contradiction. The lemma is proved. □

Remark. Assumption (G_3) directly implies that $g(t,x)$ is nonincreasing in x.

Lemma 2.2. *If assumptions (G_1), (G_2) and (G_3) hold, then*
(i) Equation (2.1) is permanent,
(ii) $\lim\limits_{t\to+\infty} |x_1(t) - x_2(t)| = 0$ for every couple of solutions $x_1(t)$ and $x_2(t)$ of (2.1) with $x_1(t_0) > 0$ and $x_2(t_0) > 0$.

Proof. (i) By (G_3), we have $\int_t^{t+\omega} g(s,x)ds = \int_t^{t+\omega} [g(s,0) + g(s,x) - g(s,0)]ds \leqslant \int_t^{t+\omega} g(s,0)ds - x \int_t^{t+\omega} a(s)ds$, and then $\limsup\limits_{t\to+\infty} \int_t^{t+\omega} g(s,x)ds \leqslant \limsup\limits_{t\to+\infty} \int_t^{t+\omega} g(s,0)ds - x \liminf\limits_{t\to+\infty} \int_t^{t+\omega} a(s)ds$. Thus, by (G_1) and (G_3), there exists positive number P such that $\limsup\limits_{t\to+\infty} \int_t^{t+\omega} g(s,P)ds < 0$. By (G_1) and (G_2), there exists positive number p $(p < P)$ such that $\liminf\limits_{t\to+\infty} \int_t^{t+\lambda} g(s,p)ds > 0$. Thus, there exist $\varepsilon > 0$ and $T \in \mathbb{R}$ such that

$$\int_t^{t+\omega} g(s,P)ds \leqslant -\varepsilon, \quad \int_t^{t+\lambda} g(s,p)ds \geqslant \varepsilon \text{ for all } t \geqslant T. \tag{2.2}$$

Claim 1. If $t_1 \geqslant T$ such that $x(t_1) = P$ and $x(t) \geqslant P$ for all $t \in [t_1, t_2]$, then $t_2 - t_1 < \omega$. Indeed, by the way of contradiction we assume that $t_2 - t_1 \geqslant \omega$, then

$$x(t_1+\omega) = x(t_1)\exp\Big\{\int\limits_{t_1}^{t_1+\omega} g(t,x(t))dt\Big\}$$
$$\leqslant x(t_1)\exp\Big\{\int\limits_{t_1}^{t_1+\omega} g(t,P)dt\Big\} \leqslant Pe^{-\varepsilon} < P,$$

which is a contradiction, since $x(t_1+\omega) \geqslant P$. The claim is proved.

Claim 2. There exists $T_1 \geqslant T$ such that $x(T_1) \leqslant P$. Indeed, suppose in the contrary that $x(t) > P$ for all $t \geqslant T$. Then $x(t) \leqslant x(T)\exp\int\limits_T^t g(s,P)ds$ for all $t \geqslant T$. Thus, (2.2) implies that $\lim\limits_{t\to+\infty} x(t) = 0$. This is a contradiction that proves the claim.

Let us put $\alpha_1 = \sup\limits_{t\in\mathbb{R}} |g(t,0)|$ and $\Delta = P\exp(\alpha_1\omega)$. By Claims 1 and 2, it follows that $x(t) \leqslant \Delta$ for all $t \geqslant T_1$.

Claim 3. If $t_1 \geqslant T$ such that $x(t_1) = p$ and $x(t) \leqslant p$ for all $t \in [t_1, t_2]$ then $t_2 - t_1 < \lambda$. Indeed, by the way of contradiction we assume that $t_2 - t_1 \geqslant \lambda$, then $x(t_1+\lambda) = x(t_1)\exp\int\limits_{t_1}^{t_1+\lambda} g(t,x(t))dt \geqslant x(t_1)\exp\int\limits_{t_1}^{t_1+\lambda} g(t,p)dt \geqslant pe^{\varepsilon} > p$, which is a contradiction, since $x(t_1+\lambda) \leqslant p$. The claim is proved.

Claim 4. There exists $T_2 \geqslant T$ such that $x(T_2) \geqslant p$. Indeed, suppose in the contrary that $x(t) < p$ for all $t \geqslant T$. Then $x(t) \geqslant x(T)\exp\int\limits_T^t g(s,p)ds$ for all $t \geqslant T$. Thus, (2.2) implies that $\lim\limits_{t\to+\infty} x(t) = +\infty$. This is a contradiction which proves the claim.

Let us put $\alpha_2 = \sup\limits_{t\in\mathbb{R}}\{|g(t,p)| + g(t,0)\}$ and $\delta = p\exp(-\alpha_2\lambda)$. By Claims 3 and 4, it follows that $x(t) \geqslant \delta$ for all $t \geqslant T_2$. The proof of part (i) is complete.
(ii) Let $x_1(t)$ and $x_2(t)$ be two arbitrary solutions of equation (2.1) with $x_1(t_0) > 0$ and $x_2(t_0) > 0$. There exist $\delta, \Delta > 0$ and $T \geqslant t_0$ such that $x_i(t) \in [\delta, \Delta]$ for all $t \geqslant T$ and $i = 1, 2$. By Lemma 2.1, without loss of generality we may assume that $x_1(t) \geqslant x_2(t)$ for all $t \geqslant T$. Let $V(t) = \ln x_1(t) - \ln x_2(t)$. Then $\dot{V}(t) = g(t,x_1(t)) - g(t,x_2(t)) \leqslant -a(t)[x_1(t) - x_2(t)] \leqslant -\delta a(t)V(t)$. Thus, $V(t) \leqslant V(T)\exp\int\limits_T^t -\delta a(s)ds \to 0$ as $t \to +\infty$. This implies $\lim\limits_{t\to+\infty} |x_1(t) - x_2(t)| = 0$. □

Lemma 2.3. *Let assumptions* (G_1), (G_2) *and* (G_3) *hold. If*

(G_4) *There exists a positive number* $\bar{\lambda}$ *such that* $\liminf_{t\to-\infty} \int_t^{t+\bar{\lambda}} g(s,0)ds > 0$ *and*

(G_5) *There exists a positive number* $\bar{\omega}$ *such that* $\liminf_{t\to-\infty} \int_t^{t+\bar{\omega}} a(s)ds > 0$,

then equation (2.1) has a unique solution $X^0(.) \in \mathcal{B}_+$.

Proof. (i) *The existence.* By the same argument as given in the proof of inequalities (2.2) in Lemma 2.2, we know that there exist $\bar{p}, \bar{P}, \bar{\varepsilon} > 0$ and $\bar{T} \in \mathbb{R}$ such that

$$\int_t^{t+\bar{\omega}} g(s,\bar{P})ds \leqslant -\bar{\varepsilon}, \qquad \int_t^{t+\bar{\lambda}} g(s,\bar{p})ds \geqslant \bar{\varepsilon} \quad \text{for all } t \leqslant \bar{T}. \tag{2.3}$$

Put $\alpha_1 = \sup_{t\in\mathbb{R}} |g(t,0)|$, $\bar{\Delta} = \bar{P}\exp(\alpha_1\bar{\omega})$, $\alpha_2 = \sup_{t\in\mathbb{R}}\{|g(t,p)| + g(t,0)\}$ and $\bar{\delta} = \bar{p}\exp(-\alpha_2\bar{\lambda})$. By the same argument as given in the proof of part (i) of Lemma 2.2, we conclude that if $x(t_0) \in [\bar{p}, \bar{P}]$ then $x(t) \in [\bar{\delta}, \bar{\Delta}]$ for all $t \in [t_0, \bar{T}]$. For each positive integer n such that $-n \leqslant \bar{T}$, let $x_n(t)$ be a solution of (2.1) with $x_n(-n) = \bar{p}$. Then $x_n(t) \in [\bar{\delta}, \bar{\Delta}]$ for all $t \in [-n, \bar{T}]$. In particular, $x_n(\bar{T}) \in [\bar{\delta}, \bar{\Delta}]$. Therefore, there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $x_{n_k}(\bar{T}) \to \xi$ as $k \to +\infty$ for some $\xi \in [\bar{\delta}, \bar{\Delta}]$. By Theorem 3.2 in [**2**, p. 14], there exist a solution $X^0(t)$ of (2.1) satisfying $X^0(\bar{T}) = \xi$ with the maximal interval of existence (ω_1, ω_1) and a subsequence $\{n_{k_j}\}$ of $\{n_k\}$ such that $x_{n_{k_j}}(t)$ converges to $X^0(t)$ uniformly on any compact subset of (ω_1, ω_2). By Lemma 2.2 (i), $\omega_2 = +\infty$. We now prove that $\omega_1 = -\infty$. To this end, by the way of contradiction we assume that $\omega_1 > -\infty$. Then there exists $t_0 \in (-\infty, \bar{T}]$ such that $X^0(t_0) \notin [\bar{\delta}, \bar{\Delta}]$. Choose a positive integer j_0 such that $-n_{k_{j_0}} < t_0$. Clearly $x_{n_{k_j}}(t_0) \in [\bar{\delta}, \bar{\Delta}]$ for all $j \geqslant j_0$ and $x_{n_{k_j}}(t_0) \to X^0(t_0)$ as $j \to +\infty$. Thus, $X^0(t_0) \in [\bar{\delta}, \bar{\Delta}]$. This is a contradiction. It implies that $\omega_1 = -\infty$. For each $\bar{t} \in (-\infty, \bar{T}]$, we know that $x_{n_{k_j}}(\bar{t}) \to X^0(\bar{t})$ as $j \to +\infty$. Thus, $X^0(\bar{t}) \in [\bar{\delta}, \bar{\Delta}]$ for all $\bar{t} \in (-\infty, \bar{T}]$. By Lemma 2.2 (i), $X^0(.) \in \mathcal{B}_+$.

(ii) *The uniqueness.* Suppose in the contrary that equation (2.1) has two distinct solutions $X^0(t)$ and $X^1(t)$ defined on $\mathbb{R}$ and satisfying $\delta \leqslant X^i(t) \leqslant \Delta$ for all $t \in \mathbb{R}$ $(i = 0, 1)$, where δ, Δ are positive constants. By Lemma 2.1, without loss of generality, we may assume that $X^0(t) \geqslant X^1(t)$ for all $t \in \mathbb{R}$. Put $V(t) = \ln X^0(t) - \ln X^1(t)$. We have $\dot{V}(t) = g(t, X^0(t)) - g(t, X^1(t)) \leqslant -a(t)[X^0(t) - X^1(t)] \leqslant -\delta a(t)V(t)$. Thus, since $V(t)$ is bounded, $0 < V(t_0) \leqslant V(t)\exp\int_t^{t_0}[-\delta a(s)]ds \to 0$ as $t \to -\infty$. This is a contradiction. The proof of Lemma 2.3 is complete. $\square$

Lemma 2.4. *Assume that*
(H_1) *For each* $i = 1, 2$, $g_i : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ *is continuous and such that the following equation*

$$\dot{x}_i = x_i g_i(t, x_i) \tag{2.4_i}$$

is permanent,
(H_2) *For each* $i = 1, 2$, *equation* (2.4_i) *has a unique solution* $X_i^0(.) \in \mathcal{B}_+$,
(H_3) *The function* $g_i(t, .)$ *is nonincreasing for each* $t \in \mathbb{R}$ *and* $g_1(t, x) \leqslant g_2(t, x)$ *for all* $(t, x) \in \mathbb{R} \times \mathbb{R}_+$.
Then $X_1^0(t) \leqslant X_2^0(t)$ *for all* $t \in \mathbb{R}$.

Proof. Suppose in the contrary that there exists $t_1 \in \mathbb{R}$ such that $X_1^0(t_1) > X_2^0(t_1)$. By (H_1), there exists a solution $\bar{x}_2(t)$ of (2.4_2) with $\bar{x}_2(t_1) = X_1^0(t_1)$ and defined on $[t_1, +\infty)$ and bounded from above and from below on $[t_1, +\infty)$ by positive constants. For $t \leqslant t_1$ let $\tilde{x}_2(t)$ be the minimal solution of (2.4_2) with $\tilde{x}_2(t_1) = X_1^0(t_1)$. By Theorem 4.1 in [2, p. 26], we have $X_1^0(t) \geqslant \tilde{x}_2(t) \geqslant X_2^0(t)$ for all $t < t_1$ in the domain of $\tilde{x}_2(t)$. Thus, $\tilde{x}_2(t)$ is defined for all $t \in (-\infty, t_1]$. Let

$$x^*(t) = \begin{cases} \bar{x}_2(t), & \text{if } t \geqslant t_1, \\ \tilde{x}_2(t), & \text{if } t < t_1. \end{cases}$$

Then $x^*(.) \in \mathcal{B}_+$. Moreover, $x^*(.)$ is a solution of (2.4_2) which is different from $X_2^0(.)$. This is a contradiction. The lemma is proved. □

Lemma 2.5. *Let hypothesis* (H_1) *hold. If*
(H_4) *There exist* $\omega > 0$ *and a function* $a : \mathbb{R} \to \mathbb{R}_+$ *which is bounded and locally integrable with* $\liminf_{t \to +\infty} \int_t^{t+\omega} a(s)ds > 0$ *such that* $D_x^+ g_1(t, x)) \leqslant -a(t)$ *for all* $(t, x) \in \mathbb{R} \times \mathbb{R}_+$,
(H_5) *For each compact set* $S \subset \mathbb{R}_+$, $\lim_{t \to +\infty} \{\sup_{x \in S} |g_1(t, x) - g_2(t, x)|\} = 0$,
then $\lim_{t \to +\infty} |x_1(t) - x_2(t)| = 0$ *for any couple of solutions* $x_1(t)$ *and* $x_2(t)$ *of equations* (2.4_1) *and* (2.4_2), *respectively, with* $x_1(t_0) > 0$ *and* $x_2(t_0) > 0$.

Proof. For each $i = 1, 2$, let $x_i(t)$ be a solution of (2.4_i) with $x_i(t_0) > 0$. By (H_1), there exist δ, $\Delta > 0$ and $T \geqslant t_0$ such that $\delta \leqslant x_i(t) \leqslant \Delta$ for all $t \geqslant T$, $i = 1, 2$. For $t \geqslant T$, let $V(t) = |\ln x_1(t) - \ln x_2(t)|$. By (H_5), we obtain

$$\begin{aligned} D^+V(t) &= [\operatorname{sign}(x_1(t) - x_2(t))] \\ &\quad \cdot \Big\{[g_1(t, x_1(t)) - g_1(t, x_2(t))] + [g_1(t, x_2(t)) - g_2(t, x_2(t))]\Big\} \\ &\leqslant -a(t)|x_1(t) - x_2(t)| + h(t) \leqslant -\delta a(t)V(t) + h(t), \end{aligned} \tag{2.5}$$

where $h(t) = |g_1(t, x_2(t)) - g_2(t, x_2(t))|$. By (H_5), we have $\lim_{t\to+\infty} h(t) = 0$. Thus, (H_4) and (2.5) imply that $\lim_{t\to+\infty} V(t) = 0$. Hence, $\lim_{t\to+\infty} |x_1(t) - x_2(t)| = 0$. □

Consider the following equation

$$\dot{y} = f(t, y), \tag{2.6}$$

where $f : \mathbb{R} \times \Omega \to \mathbb{R}^d$ ($\Omega \subset \mathbb{R}^d$ is open) is almost periodic in t uniformly for $y \in \Omega$. We recall Bochner's criterion for the almost periodicity (see [**8**]): *$f(t, y)$ is almost periodic in t uniformly for $y \in \Omega$ if and only if for every sequence of numbers $\{\tau_k\}_{k=1}^{\infty}$, there exists a subsequence $\{\tau_{k_l}\}_{l=1}^{\infty}$ such that the sequence of translations $\{f(\tau_{k_l} + t, y)\}_{l=1}^{\infty}$ converges uniformly on $\mathbb{R} \times S$, where S is any compact subset of Ω.*

Denote by f_τ the τ-translation of f, that is $f_\tau(t, y) = f(\tau + t, y)$; $H(f)$ the hull of f, that is the closure of $\{f_\tau : \tau \in \mathbb{R}\}$ in the topology of uniform convergence on compact subsets of $\mathbb{R} \times \Omega$. We know that $H(f)$ is compact and for $f^* \in H(f)$, $f^*(t, y)$ is almost periodic in t uniformly for $y \in \Omega$. Denote by $\mathcal{C}$ the set of continuous functions from $\mathbb{R} \times \Omega$ into $\mathbb{R}^d$ equipped with the topology of uniform convergence on compact subsets of $\mathbb{R} \times \Omega$.

Lemma 2.6. *Let S be a compact subset of Ω. Assume that for each $f^* \in H(f)$, the following equation*

$$\dot{y} = f^*(t, y) \tag{2.7}$$

has a unique solution $y^*(t)$ which is defined on whole $\mathbb{R}$ and $y^*(t) \in S$ for all $t \in \mathbb{R}$. Then equation (2.6) has a unique almost periodic solution in S and its module is contained in the module of $f(t, y)$.

Proof. Let $y_0(t)$ be the unique solution of (2.6) with $y_0(t) \in S$ for all $t \in \mathbb{R}$. Let $\{\tau_k\}_{k=1}^{\infty}$ be a sequence such that $f_{\tau_k} \to f^*$ as $k \to \infty$ uniformly on $\mathbb{R} \times K$, where K is any compact subset of Ω. We claim that $y_0(\tau_k + t) \to y^*(t)$ as $k \to \infty$ uniformly on $\mathbb{R}$, where $y^*(t)$ is the unique solution of (2.7) with $y^*(t) \in S$ for all $t \in \mathbb{R}$. To this end, by the way of contradiction we assume that there exist a subsequence $\{\tau_{k_l}\}_{l=1}^{\infty}$ of $\{\tau_k\}_{k=1}^{\infty}$, a sequence of numbers $\{s_l\}_{l=1}^{\infty}$ and a positive number α such that $\|y_0(s_l + \tau_{k_l}) - y^*(s_l)\| \geqslant \alpha$ for all l. By Bochner's criterion, we may assume, without loss of generality, that $f_{\tau_{m_l}+s_l} \to \hat{f}$ as $l \to \infty$ uniformly on $\mathbb{R} \times K$, where K is any compact subset of Ω. Thus, $f^*_{s_l} \to \hat{f}$ as $l \to \infty$ uniformly on $\mathbb{R} \times K$, where K is any compact subset of Ω. Since S is compact, we may without loss of generality assume that $y_0(\tau_{k_l} + s_l) \to \xi_0$ and $y^*(s_l) \to \xi^*$ as $l \to \infty$. We know that ξ_0, $\xi^* \in S$ and $\|\xi_0 - \xi^*\| \geqslant \alpha$. It is clear that $y_0(t + \tau_{k_l} + s_l)$ is a solution of the following equation

$$\dot{y} = f(t + \tau_{k_l} + s_l, y). \tag{2.8$_l$}$$

Consider the following equation

$$\dot{y} = \hat{f}(t, y). \tag{2.9}$$

Now $f_{\tau_{k_l}+s_l} \to \hat{f}$ uniformly on any compact subset of $\mathbb{R} \times \Omega$ as $l \to \infty$, Theorem 3.2 in [**2**, p. 14] shows that there exist a solution $y(t)$ of (2.9) with $y(0) = \xi_0$ having a maximal interval of existence (ω_1, ω_2) and a subsequence of $\{\tau_{k_l} + s_l\}_{l=1}^{\infty}$ therefore, without loss of generality, we may assume that there is $\{\tau_{k_l} + s_l\}_{l=1}^{\infty}$ such that $y_0(t + \tau_{k_l} + s_l) \to y(t)$ uniformly on any compact subset of (ω_1, ω_2) as $l \to \infty$. Since S is compact, Theorem 3.1 in [**2**, p. 12] shows that $\omega_1 = -\infty$ and $\omega_2 = +\infty$. Thus, $y(t) \in S$ for all $t \in \mathbb{R}$.

We know that $y^*(t + s_l)$ is a solution of the following equation

$$\dot{y} = f^*(t + s_k, y). \tag{2.10}$$

By the same argument as given above, there exists a solution $\bar{y}(t)$ of (2.10) with $\bar{y}(0) = \xi^*$ and $\bar{y}(t) \in S$ for all $t \in \mathbb{R}$. By the uniqueness of solution of (2.10) defined on $\mathbb{R}$ and contained in S, we have $y(t) = \bar{y}(t)$ for all $t \in \mathbb{R}$. Thus, $\xi_0 = y(0) = \bar{y}(0) = \xi^*$, but this contradicts $\|\xi_0 - \xi^*\| \geqslant \alpha$. The claim is proved. By Bochner's criterion, $y_0(t)$ is almost periodic.

By the module containment theorem [**8**, p. 18], the module of $y_0(t)$ is contained in the module of $f(t, y)$. □

Lemma 2.7. *Assume that $g(t,x)$ is almost periodic in t uniformly for $x \in \mathbb{R} \times \mathbb{R}_+$ and*

(G_1^*) $\lim\limits_{T\to+\infty} \dfrac{1}{T} \displaystyle\int_0^T g(s,0)ds > 0,$

(G_2^*) *There exists an almost periodic function $a : \mathbb{R} \to \mathbb{R}_+$ such that*

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T a(s)ds > 0 \textit{ and } D_x^+ g(t,x)) \leqslant -a(t) \textit{ for all } (t,x) \in \mathbb{R} \times \mathbb{R}_+.$$

Then equation (2.1) has a unique solution $X^0(.) \in \mathcal{B}_+$. Moreover, $X^0(.)$ is almost periodic, its module is contained in the module of $g(t,x)$ and $\lim\limits_{t\to+\infty} |x(t) - X^0(t)| = 0$ for any solution $x(t)$ of (2.1) with $x(t_0) > 0$. In particular, if $g(t,x)$ is Θ-periodic in t $(\Theta > 0)$, then also the solution $X^0(t)$ is Θ-periodic.

Proof. By almost periodicity, (G_1^*) and (G_2^*) imply that there exist positve numbers λ and γ such that $\displaystyle\int_t^{t+\lambda} g(s,0)ds > \gamma$ and $\displaystyle\int_t^{t+\lambda} a(s)ds > \gamma$ for all $t \in \mathbb{R}$. By the same argument as given in the proof of inequalities (2.2) of Lemma 2.2, there exist positive numbers p, P and ε such that

$$\int_t^{\lambda+t} g(s,P)ds \leqslant -\varepsilon, \quad \int_t^{\lambda+t} g(s,p)ds \geqslant \varepsilon \quad \text{for all} \quad t \in \mathbb{R}. \tag{2.11}$$

By almost periodicity of $g(t,x)$, it is easy to see that

$$\int_t^{\lambda+t} g^*(s,P)ds \leqslant -\varepsilon, \quad \int_t^{\lambda+t} g^*(s,p)ds \geqslant \varepsilon, \quad \text{for all} \quad t \in \mathbb{R} \text{ and } g^* \in H(g). \tag{2.12}$$

Put $\alpha_1 = \sup_{t\in\mathbb{R}} |g^*(t,0)|$, $\Delta = P\exp(\alpha_1\lambda)$, $\alpha_2 = \sup_{t\in\mathbb{R}}\{|g^*(t,p)| + g^*(t,0)\}$ and $\delta = p\exp(-\alpha_2\lambda)$. It is easy to see that δ and Δ do not depend on the choice of $g^* \in H(g)$.

Let $g^* \in H(g)$; consider the following equation

$$\dot{x} = xg^*(t,x). \tag{2.13}$$

By the same argument as given in the proof of Lemma 2.3, we can show that (2.13) has a unique solution $X^*(t)$ defined on $\mathbb{R}$ with $X^*(t) \in [\delta, \Delta]$ for all $t \in \mathbb{R}$. It follows from Lemmas 2.2 and 2.6 that equation (2.1) has a unique almost periodic solution $X^0(.) \in \mathcal{B}_+$, which satisfies $\lim_{t\to+\infty} |x(t) - X^0(t)| = 0$ for any solution $x(t)$ of equation (2.1) with $x(t_0) > 0$ and its module is contained in that of $g(t,x)$. If g is Θ-periodic in t, then $X^0(.)$, $X^0_\Theta(.) \in \mathcal{B}_+$ are two solutions of equation (2.1). By the uniqueness, $X^0(.) \equiv X^0_\Theta(.)$. The lemma is proved. □

3. Permanence and bounded solutions of Kolmogorov predator--prey system. Consider the following Kolmogorov predator-prey system

$$\begin{aligned} \dot{u}_i &= u_i f_i(t, u_1, \ldots, u_n, v_1, \ldots, v_m), \ i = 1, \ \ldots, \ n, \\ \dot{v}_j &= v_j h_j(t, u_1, \ldots, u_n, v_1, \ldots, v_m), \ j = 1, \ \ldots, \ m, \end{aligned} \tag{3.1}$$

where $f_i, h_j : \mathbb{R} \times \mathbb{R}^{n+m}_+ \to \mathbb{R}$ are continuous. For $w, z \in \mathbb{R}^d$, we set $w \leqslant z$ if $w_i \leqslant z_i$, $i = 1, \ldots, d$. Let $\mathcal{B}^d_+ = \{(\phi_1, \ldots, \phi_d) : \mathbb{R} \to \mathbb{R}^d \mid \phi_i \in \mathcal{B}_+, \ i = 1, \ldots, d\}$. We introduce the following hypotheses:

(K_1) f_i, h_j are bounded on any set of the form $\mathbb{R} \times S$, where $S \subset \mathbb{R}^{n+m}_+$ is compact, and are such that for each compact set $S \subset \mathbb{R}^{n+m}_+$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f_i(t,u,v) - f_i(t,\bar{u},\bar{v})| < \varepsilon$, $|h_j(t,u,v) - h_j(t,\bar{u},\bar{v})| < \varepsilon$ for all $t \in \mathbb{R}$, $i = 1, \ldots, n$, $j = 1, \ldots, m$ and $(u,v), (\bar{u},\bar{v}) \in S$ with $\|(u,v) - (\bar{u},\bar{v})\| < \delta$.

(K_2) For each $i = 1, \ldots, n$, there exist positive numbers λ_i^+ and λ_i^- such that

$$\liminf_{t\to+\infty} \int_t^{t+\lambda_i^+} f_i(s, 0, \ldots, 0)ds > 0, \ \liminf_{t\to-\infty} \int_t^{t+\lambda_i^-} f_i(s, 0, \ldots, 0)ds > 0,$$

(K_3) For each $i = 1, \ldots, n$, there exist positive numbers ω_i^+, ω_i^- and a bounded locally integrable function $a_i : \mathbb{R} \to \mathbb{R}_+$ with

$$\liminf_{t\to+\infty} \int_t^{t+\omega_i^+} a_i(s)ds > 0 \quad \text{and} \quad \liminf_{t\to-\infty} \int_t^{t+\omega_i^-} a_i(s)ds > 0$$

such that $D^+_{u_i} f_i(t, u, v)) \leqslant -a_i(t)$ for $(t, u, v) \in \mathbb{R} \times \mathbb{R}_+^{n+m}$,

(K_4) For each $j = 1, \ldots, m$, there exist positive numbers γ_j^+, γ_j^- and a bounded locally integrable function $e_j : \mathbb{R} \to \mathbb{R}_+$ with

$$\liminf_{t\to+\infty} \int_t^{t+\gamma_j^+} e_j(s)ds > 0 \quad \text{and} \quad \liminf_{t\to-\infty} \int_t^{t+\gamma_j^-} e_j(s)ds > 0$$

such that $D^+_{v_j} h_j(t, u, v)) \leqslant -e_j(t)$ for $(t, u, v) \in \mathbb{R} \times \mathbb{R}_+^{n+m}$,

(K_5) For each $i = 1, \ldots, n$, $f_i(t, u_1, \ldots, u_n, v_1, \ldots, v_m)$ is nonincreasing in each variable u_l for $l = 1, \ldots, n$ and in each variable v_k for $k = 1, \ldots, m$,

(K_6) For each $j = 1, \ldots, m$, $h_j(t, u_1, \ldots, u_n, v_1, \ldots, v_m)$ is nondecreasing in each variable u_l for $l = 1, \ldots, n$ and is nonincreasing in each variable v_k for $k = 1, \ldots, m$.

Note that by (K_1), (K_2), (K_3) and Lemma 2.3, for each $i = 1, \ldots, n$, the following equation

$$(3.2_i) \qquad \dot{u}_i = u_i f_i(t, 0, \ldots, 0, u_i, 0, \ldots, 0)$$

has a unique solution $U_i^0(.) \in \mathcal{B}_+$. Put $U^0(t) = (U_1^0(t), \ldots, U_n^0(t))$.

(K_7) For each $j = 1, \ldots, m$, there exist positive numbers μ_j^+, μ_j^- such that

$$\liminf_{t\to+\infty} \int_t^{t+\mu_j^+} h_j(s, U^0(s), 0, \ldots, 0)ds > 0, \ \liminf_{t\to-\infty} \int_t^{t+\mu_j^-} h_j(s, U^0(s), 0, \ldots, 0)ds > 0.$$

Note that by (K_1), (K_4), (K_7) and Lemma 2.3, for each $j = 1, \ldots, m$, the following equation

$$(3.3_j) \qquad \dot{v}_j = v_j h_j(t, U^0(t), 0, \ldots, 0, v_j, 0, \ldots, 0)$$

has a unique solution $V_j^0(.) \in \mathcal{B}_+$. Put $V^0(t) = (V_1^0(t), \ldots, V_m^0(t))$.

(K_8) For each $i = 1, \dots, n$, there exist positive numbers $\nu_i^+,\ \nu_i^-$ such that

$$\liminf_{t\to+\infty} \int_t^{t+\nu_i^+} f_i(s, U_1^0(s), \dots, U_{i-1}^0(s), 0, U_{i+1}^0(s), \dots, U_n^0(s), V^0(s))ds > 0,$$

$$\liminf_{t\to-\infty} \int_t^{t+\nu_i^-} f_i(s, U_1^0(s), \dots, U_{i-1}^0(s), 0, U_{i+1}^0(s), \dots, U_n^0(s), V^0(s))ds > 0.$$

Note that by (K_1), (K_3), (K_8) and Lemma 2.3, for each $i = 1, \dots, n$, the following equation

$$(3.4_i) \qquad \dot{u}_i = u_i f_i(t, U_1^0(t), \dots, U_{i-1}^0(t), u_i, U_{i+1}^0(t), \dots, U_n^0(t), V^0(t))$$

has a unique solution $u_i^0(.) \in \mathcal{B}_+$. Put $u^0(t) = (u_1^0(t), \dots, u_n^0(t))$.

(K_9) For each $j = 1, \dots, m$, there exist positive numbers $\varepsilon_j^+,\ \varepsilon_j^-$ such that

$$\liminf_{t\to+\infty} \int_t^{t+\varepsilon_j^+} h_j(s, u^0(s), V_1^0(s), \dots, V_{j-1}^0(s), 0, V_{j+1}^0(s), \dots, V_m^0(s))ds > 0,$$

$$\liminf_{t\to-\infty} \int_t^{t+\varepsilon_j^-} h_j(s, u^0(s), V_1^0(s), \dots, V_{j-1}^0(s), 0, V_{j+1}^0(s), \dots, V_m^0(s))ds > 0.$$

Note that by (K_1), (K_4), (K_9) and Lemma 2.3, for each $j = 1, \dots, m$, the following equation

$$(3.5_j) \qquad \dot{v}_j = v_j h_j(t, u^0(t), V_1^0(t), \dots, V_{j-1}^0(t), v_j, V_{j+1}^0(t), \dots, V_m^0(t))$$

has a unique solution $v_j^0(.) \in \mathcal{B}_+$. Put $v^0(t) = (v_1^0(t), \dots, v_m^0(t))$.

THEOREM 3.1. *Let (K_1)–(K_9) hold. Then system (3.1) is permanent and it has at least one solution $(u^*(.), v^*(.)) \in \mathcal{B}_+^{n+m}$.*

PROOF. (*i*) *The existence.* By Lemma 2.4, $(u^0(t), v^0(t)) \leqslant (U^0(t), V^0(t))$ for all $t \in \mathbb{R}$. We denote by $\mathcal{C}$ the set of continuous functions $(u(.), v(.)) : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^m$ equipped with the topology of uniform convergence on compact subsets of $\mathbb{R}$. It is well-known that $\mathcal{C}$ is a Fréchet space. Let

$$\mathcal{M} = \{(u(.), v(.)) \in \mathcal{C} : (u^0(t), v^0(t)) \leqslant (u(t), v(t)) \leqslant (U^0(t), V^0(t)) \text{ for all } t \in \mathbb{R}\}.$$

By (K_1), (K_3), (K_4), (K_8) and (K_9), Lemma 2.3 implies that for each $(\tilde{u}(.), \tilde{v}(.)) \in \mathcal{M}$, the following system of $n+m$ uncoupled differential equations

$$\begin{aligned} \dot{u}_i &= u_i f_i(t, \tilde{u}_1(t), \ldots, \tilde{u}_{i-1}(t), u_i, \tilde{u}_{i+1}(t), \ldots, \tilde{u}_n(t), \tilde{v}(t)),\ i{=}1, \ldots, n, \\ \dot{v}_j &= v_j h_j(t, \tilde{u}(t), \tilde{v}_1(t), \ldots, \tilde{v}_{j-1}(t), v_j, \tilde{v}_{j+1}(t), \ldots, \tilde{v}_m(t)),\ j{=}1, \ldots, m, \end{aligned} \tag{3.6}$$

has a unique solution $(\bar{u}(.), \bar{v}(.)) \in \mathcal{B}_+^{n+m}$. By Lemma 2.4, $(u^0(t), v^0(t)) \leqslant (\bar{u}(t), \bar{v}(t)) \leqslant (U^0(t), V^0(t))$ for all $t \in \mathbb{R}$. Hence, we can introduce the following operator

$$\mathcal{T} : \mathcal{M} \to \mathcal{M},\ (\tilde{u}(.), \tilde{v}(.)) \mapsto (\bar{u}(.), \bar{v}(.)).$$

Clearly, $(u^*(.), v^*(.))$ is a solution in $\mathcal{M}$ of system (3.1) if and only if it is a fixed point of $\mathcal{T}$. Let

$$\begin{aligned} \delta &= \inf\{u_i^0(t), v_j^0(t) \colon i = 1, \ldots, n,\ j = 1, \ldots, m,\ t \in \mathbb{R}\}, \\ \Delta &= \sup\{U_i^0(t), V_j^0(t) \colon i = 1, \ldots, n,\ j = 1, \ldots, m,\ t \in \mathbb{R}\}, \\ L &= \sup\{|u_i f_i(t, u, v)|,\ |v_j h_j(t, u, v)| \colon i = 1, \ldots, n,\ j = 1, \ldots, m, \\ &\qquad\qquad (t, u, v) \in \mathbb{R} \times [\delta, \Delta]^{n+m}\}. \end{aligned}$$

By (K_1), $0 < L < +\infty$. Let us set

$$\mathcal{M}_1 = \{\phi \in \mathcal{M} : |\phi_i(t) - \phi_i(\bar{t})| \leqslant L|t - \bar{t}|,\ i = 1, \ldots, n+m,\ t, \bar{t} \in \mathbb{R}\}.$$

It is easily seen that $\mathcal{M}_1$ is a closed convex subset of $\mathcal{M}$. By Ascoli's theorem (see [**4**]), $\mathcal{M}_1$ is compact (in the topology of uniform convergence on compact subsets of $\mathbb{R}$). Moreover, $\mathcal{T}(\mathcal{M}_1) \subset \mathcal{M}_1$.

Claim. The operator $\mathcal{T}$ is continuous on $\mathcal{M}_1$ in the topology of uniform convergence on compact subsets of $\mathbb{R}$. To prove this, let $\{(u^k(.), v^k(.))\}_{k=1}^{\infty} \subset \mathcal{M}_1$ such that $(u^k(.), v^k(.)) \to (\tilde{u}(.), \tilde{v}(.))$ as $k \to +\infty$. Since $\mathcal{M}_1$ is closed, $(\tilde{u}(.), \tilde{v}(.)) \in \mathcal{M}_1$. We shall show that $\mathcal{T}(u^k(.), v^k(.)) \to \mathcal{T}(\tilde{u}(.), \tilde{v}(.))$ as $t \to +\infty$. Since $\{\mathcal{T}(u^k(.), v^k(.))\}_{k=1}^{\infty}$ is precompact, it suffices to show that if a subsequence $\{\mathcal{T}(u^{k_s}(.), v^{k_s}(.))\}$ converges to $(\bar{u}(.), \bar{v}(.))$ then $(\bar{u}(.), \bar{v}(.)) = \mathcal{T}(\tilde{u}(.), \tilde{v}(.))$. To this end, let us consider two systems

$$\begin{cases} \dot{u}_i = u_i f_i(t, u_1^{k_s}(t), \ldots, u_{i-1}^{k_s}(t), u_i, u_{i+1}^{k_s}(t), \ldots, u_n^{k_s}(t), v^{k_s}(t)),\ i{=}1, \ldots, n, \\ \dot{v}_j = v_j h_j(t, u^{k_s}(t), v_1^{k_s}(t), \ldots, v_{j-1}^{k_s}(t), v_j, v_{j+1}^{k_s}(t), \ldots, v_m^{k_s}(t)),\ j{=}1, \ldots, m, \end{cases} \tag{3.7$_{k_s}$}$$

and

$$\begin{cases} \dot{u}_i = u_i f_i(t, \tilde{u}_1(t), \ldots, \tilde{u}_{i-1}(t), u_i, \tilde{u}_{i+1}(t), \ldots, \tilde{u}_n(t), \tilde{v}(t)),\ i{=}1, \ldots, n, \\ \dot{v}_j = v_j h_j(t, \tilde{u}(t), \tilde{v}_1(t), \ldots, \tilde{v}_{j-1}(t), v_j, \tilde{v}_{j+1}(t), \ldots, \tilde{v}_m(t)),\ j{=}1, \ldots, m. \end{cases} \tag{3.8}$$

Clearly, the right hand side of (3.7_{k_s}) converges to the right hand side of (3.8) uniformly on any compact subset of $\mathbb{R} \times \mathbb{R}_+^{n+m}$. By Theorem 2.4 in [**2**, p. 4], it follows that $(\bar{u}(.), \bar{v}(.))$ is a solution of (3.8). Since (3.8) has a unique solution in $\mathcal{M}$ (by Lemma 2.3), $\mathcal{T}(\tilde{u}(.), \tilde{v}(.)) = (\bar{u}(.), \bar{v}(.))$. The claim is proved.

By Tychonov's fixed point theorem (see [**1**]), there exists $(u^*(.), v^*(.)) \in \mathcal{M}_1$ such that $\mathcal{T}(u^*(.), v^*(.)) = (u^*(.), v^*(.))$. Thus, $(u^*(.), v^*(.))$ is a solution of system (3.1).

(ii) The permanence. Let $(u(t), v(t))$ be a solution of (3.1) with $(u_i(t_0), v_j(t_0)) \in \operatorname{int} \mathbb{R}_+^{n+m}$. For each $i = 1, \ldots, n$, let $\bar{u}_i(t)$ be a solution of (3.2_i) with $\bar{u}_i(t_0) = u_i(t_0)$. By Lemma 2.1 and the comparison theorem,

$$\bar{u}_i(t) \geqslant u_i(t) \text{ for all } t \geqslant t_0,\ i = 1, \ldots, n. \tag{3.9}$$

By Lemma 2.2,

$$\lim_{t\to+\infty} |\bar{u}_i(t) - U_i^0(t)| = 0 \text{ for } i = 1, \ldots, n. \tag{3.10}$$

From (3.9) and (3.10), we have

$$\limsup_{t\to+\infty} u_i(t) \leqslant \limsup_{t\to+\infty} U_i^0(t) \leqslant \Delta \quad \text{for } i = 1, \ldots, n. \tag{3.11}$$

For each $j = 1, \ldots, m$, let $\bar{v}_j(t)$ be a solution with $\bar{v}_j(t_0) = v_j(t_0)$ of the following equation

$$\dot{v}_j = v_j h_j(t, \bar{u}(t), 0, \ldots, 0, v_j, 0, \ldots, 0). \tag{3.12_j}$$

By (3.10), (K_1), (K_4) and (K_7), we can apply Lemma 2.5 to equations (3.3_j) and (3.12_j) and obtain

$$\lim_{t\to+\infty} |\bar{v}_j(t) - V_j^0(t)| = 0 \text{ for } j = 1, \ldots, m. \tag{3.13}$$

By Lemma 2.1 and the comparison theorem,

$$\bar{v}_j(t) \geqslant v_j(t) \text{ for all } t \geqslant t_0,\ j = 1, \ldots, m. \tag{3.14}$$

From (3.13) and (3.14), we have

$$\limsup_{t\to+\infty} v_j(t) \leqslant \limsup_{t\to+\infty} V_j^0(t) \leqslant \Delta \quad \text{for } j = 1, \ldots, m. \tag{3.15}$$

For $i = 1, \ldots, n$, let $\tilde{u}_i(t)$ be a solution with $\tilde{u}_i(t_0) = u_i(t_0)$ of the following equation

$$\dot{u}_i = u_i f_i(t, \bar{u}_1(t), \ldots, \bar{u}_{i-1}(t), u_i, \bar{u}_{i+1}(t), \ldots, \bar{u}_n(t), \bar{v}(t)). \tag{3.16_i}$$

By (3.10), (3.13), (K_1), (K_3) and (K_8), we can apply Lemma 2.5 to equations (3.4_i) and (3.16_i) and obtain

$$\lim_{t\to+\infty} |\tilde{u}_i(t) - u_i^0(t)| = 0 \text{ for } i = 1, \ldots, n. \tag{3.17}$$

By Lemma 2.1 and the comparison theorem,

$$u_i(t) \geqslant \tilde{u}_i(t) \text{ for all } t \geqslant t_0,\ i = 1, \ldots, n. \tag{3.18}$$

From (3.17) and (3.18) we have

$$\liminf_{t\to+\infty} u_i(t) \geqslant \liminf_{t\to+\infty} u_i^0(t) \geqslant \delta \quad \text{for } i = 1, \dots n. \tag{3.19}$$

For each $j = 1, \dots m$, let $\tilde{v}_j(t)$ be a solution with $\tilde{v}_j(t_0) = v_j(t_0)$ of the following equation

$$\dot{v}_j = v_j h_j(t, \tilde{u}(t), \bar{v}_1(t), \dots, \bar{v}_{j-1}(t), v_j, \bar{v}_{j+1}(t), \dots, \bar{v}_m(t)). \tag{3.20_j}$$

By (3.13), (3.17), (K_1), (K_4) and (K_9), we can apply Lemma 2.5 to equations (3.5_j) and (3.20_j) and obtain

$$\lim_{t\to+\infty} |\tilde{v}_j(t) - v_j^0(t)| = 0 \text{ for } j = 1, \dots, m. \tag{3.21}$$

By Lemma 2.1 and the comparison theorem,

$$v_j(t) \geqslant \tilde{v}_j(t) \text{ for all } t \geqslant t_0,\ j = 1, \dots, m. \tag{3.22}$$

From (3.21) and (3.22) we have

$$\liminf_{t\to+\infty} v_j(t) \geqslant \liminf_{t\to+\infty} v_j^0(t) \geqslant \delta \quad \text{for } j = 1, \dots m. \tag{3.23}$$

By (3.11), (3.15), (3.19) and (3.23), system (3.1) is permanent. □

Remark. Theorem 3.1 is an extension of Theorem 1 in [**5**] to system (3.1). It is also an extension of Theorem 2.5 in [**6**] to the nonperiodic case.

Using Theorem 3.1, we have the following corollary:

Corollary 3.2. *Assume that f_i, h_j $(i = 1, \dots, n,\ j = 1, \dots, m)$ are almost periodic in t uniformly for $(u, v) \in \mathbb{R}_+^{n+m}$ and satisfy (K_5), (K_6) and the following hypotheses:*

(K_2^*) $\lim\limits_{T\to+\infty} \frac{1}{T}\int_0^T f_i(t, 0, \dots, 0)dt > 0$ *for $i = 1, \dots, n$,*

(K_3^*) *For each $i = 1, \dots, n$, there exists a nonnegative almost periodic function $a_i(t)$ with $\lim\limits_{T\to+\infty} \frac{1}{T}\int_0^T a_i(t)dt > 0$ such that $D_{u_i}^+ f_i(t, u, v)) \leqslant -a_i(t)$ for $(t, u, v) \in \mathbb{R} \times \mathbb{R}_+^{n+m}$,*

(K_4^*) *For each $j = 1, \dots, m$, there exists a nonnegative almost periodic function $e_j(t)$ with $\lim\limits_{T\to+\infty} \frac{1}{T}\int_0^T e_j(t)dt > 0$ such that $D_{v_j}^+ h_j(t, u, v)) \leqslant -e_j(t)$ for $(t, u, v) \in \mathbb{R} \times \mathbb{R}_+^{n+m}$,*

(K_7^*) $\lim\limits_{T\to+\infty} \frac{1}{T}\int\limits_0^T h_j(t, U^0(t), 0, \ldots, 0)dt > 0$ *for* $j = 1, \ldots, m$,

(K_8^*) $\lim\limits_{T\to+\infty} \frac{1}{T}\int\limits_0^T f_i(t, U_1^0(t), \ldots, U_{i-1}^0(t), 0, U_{i+1}^0(t), \ldots, U_n^0(t), V^0(t))dt > 0$ *for* $i = 1, \ldots, n$,

(K_9^*) $\lim\limits_{T\to+\infty} \frac{1}{T}\int\limits_0^T h_j(t, u^0(t), V_1^0(t), \ldots, V_{j-1}^0(t), 0, V_{j+1}^0(t), \ldots, V_m^0(t))dt > 0$ *for* $j = 1, \ldots, m$.

Then system (3.1) is permanent and it has at least one solution $(u^*(.), v^*(.)) \in \mathcal{B}_+^{n+m}$. *In particular, if* f_i, h_j $(i = 1, \ldots, n,\ j = 1, \ldots, m)$ *are* Θ*-periodic* $(\Theta > 0)$ *in* t, *then system (3.1) has least one* Θ*-periodic solution* $(u^*(.), v^*(.)) \in \mathcal{B}_+^{n+m}$.

4. Lotka–Volterra predator-prey system. Consider the following Lotka–Volterra predator-prey system

$$
\begin{aligned}
\dot{u}_i &= u_i\Big[b_i(t) - \sum_{k=1}^{n} a_{ik}(t)u_k - \sum_{k=1}^{m} c_{ik}(t)v_k\Big],\ i = 1, \ldots, n,\\
\dot{v}_j &= v_j\Big[r_j(t) + \sum_{k=1}^{n} d_{jk}(t)u_k - \sum_{k=1}^{m} e_{jk}(t)v_k\Big],\ j = 1, \ldots, m,
\end{aligned}
\tag{4.1}
$$

where $a_{ik}(t),\ c_{ik}(t),\ d_{jk}(t),\ e_{jk}(t)$ are continuous, nonnegative and bounded on $\mathbb{R}$, $b_i(t),\ r_j(t)$ are continuous and bounded on $\mathbb{R}$. We introduce the following hypotheses:

(L_1) For each $i = 1, \ldots, n$, there exist positive numbers λ_i^+ and λ_i^- such that

$$\liminf_{t\to+\infty} \int\limits_t^{t+\lambda_i^+} b_i(s)ds > 0, \quad \liminf_{t\to-\infty} \int\limits_t^{t+\lambda_i^-} b_i(s)ds > 0,$$

(L_2) For each $i = 1, \ldots, n$, there exist positive numbers ω_i^+ and ω_i^- such that

$$\liminf_{t\to+\infty} \int\limits_t^{t+\omega_i^+} a_{ii}(s)ds > 0, \quad \liminf_{t\to-\infty} \int\limits_t^{t+\omega_i^-} a_{ii}(s)ds > 0,$$

(L_3) For each $j = 1, \dots, m$, there exist positive numbers γ_j^+ and γ_j^- such that

$$\liminf_{t\to+\infty} \int\limits_t^{t+\gamma_j^+} e_{jj}(s)ds > 0, \quad \liminf_{t\to-\infty} \int\limits_t^{t+\gamma_j^-} e_{jj}(s)ds > 0,$$

(L_4) For each $i = 1, \dots, n$, there exist positive numbers $\mu_j^+,\ \mu_j^-$ such that

$$\liminf_{t\to+\infty} \int\limits_t^{t+\mu_j^+} \Big[r_j(s) + \sum_{k=1}^m d_{jk}(s)U_k^0(s)\Big]ds > 0,$$
$$\liminf_{t\to-\infty} \int\limits_t^{t+\mu_j^-} [r_j(s) + \sum_{k=1}^m d_{jk}(s)U_k^0(s)]ds > 0,$$

where $U_i^0(.)$ is a unique solution in $\mathcal{B}_+$ of the following equation

$$(4.2_i) \qquad \dot{u}_i = u_i[b_i(t) - a_{ii}(t)u_i].$$

(L_5) For each $i = 1, \dots, n$, there exist positive numbers ν_i^+ and ν_i^- such that

$$\liminf_{t\to+\infty} \int\limits_t^{t+\nu_i^+} \Big[b_i(s) - \sum_{k=1,\ k\neq i}^n a_{ik}(s)U_k^0(s) - \sum_{k=1}^m c_{ik}(s)V_k^0(s)\Big]ds > 0,$$
$$\liminf_{t\to-\infty} \int\limits_t^{t+\nu_i^-} \Big[b_i(s) - \sum_{k=1,\ k\neq i}^n a_{ik}(s)U_k^0(s) - \sum_{k=1}^m c_{ik}(s)V_k^0(s)\Big]ds > 0,$$

where $V_j^0(.)$ is a unique solution in $\mathcal{B}_+$ of the following equation

$$(4.3_j) \qquad \dot{v}_j = v_j\Big[r_j(t) + \sum_{k=1}^m d_{jk}(t)U_k^0(t) - e_{jj}(t)v_j\Big],$$

(L_6) For each $j = 1, \dots, m$, there exist positive numbers ε_j^+ and ε_j^- such that

$$\liminf_{t\to+\infty} \int\limits_t^{t+\varepsilon_j^+} \Big[r_j(s) + \sum_{k=1}^m d_{jk}(s)u_k^0(s) - \sum_{k=1,\ k\neq j}^m e_{jk}(s)V_k^0(s)\Big]ds > 0,$$
$$\liminf_{t\to-\infty} \int\limits_t^{t+\varepsilon_j^-} \Big[r_j(s) + \sum_{k=1}^m d_{jk}(s)u_k^0(s) - \sum_{k=1,\ k\neq j}^m e_{jk}(s)V_k^0(s)\Big]ds > 0,$$

where $u_i^0(.)$ is the unique solution in $\mathcal{B}_+$ of the following equation

$$(4.4_i) \qquad \dot{u}_i = u_i\Big[b_i(t) - \sum_{k=1,\ k\neq i}^{n} a_{ik}(t)U_k^0(t) - \sum_{k=1}^{m} c_{ik}(t)V_k^0(t) - a_{ii}(t)u_i\Big].$$

Applying Theorem 3.1 to system (4.1) we obtain the following corollary:

Corollary 4.1. *Let* (L_1)*–*(L_6) *hold. Then system (4.1) is permanent and it has at least one solution* $(u^*(.), v^*(.)) \in \mathcal{B}_+^{n+m}$*.*

Definition. A solution $(\bar{u}(t), \bar{v}(t))$ of (3.1) with $(\bar{u}(t_0), \bar{v}(t_0)) \in \operatorname{int}\mathbb{R}_+^{n+m}$ is said to be globally attractive, if for any solution $(u(t), v(t))$ with $(u(t_0), v(t_0)) \in \operatorname{int}\mathbb{R}_+^{n+m}$ there is $\lim\limits_{t\to+\infty} \|(u(t), v(t)) - (\bar{u}(t), \bar{v}(t))\| = 0$.

Theorem 4.2. *Let* (L_1)*–*(L_6) *hold. If*
(L_7) *There exist positive numbers* s_i, β_j $(i = 1, \dots, n,\ j = 1, \dots, m)$ *and a continuous nonnegative function* $\alpha : \mathbb{R} \to \mathbb{R}$ *with* $\int\limits_0^{+\infty} \alpha(t)dt = +\infty$, $\int\limits_{-\infty}^{0} \alpha(t)dt = +\infty$ *such that*

$$s_i a_{ii}(t) - \sum_{k=1,\ k\neq i}^{n} s_k a_{ki}(t) - \sum_{k=1}^{m} \beta_k d_{ki}(t) \geqslant \alpha(t) \quad \text{for all } t \in \mathbb{R},\ i = 1, \dots, n,$$

$$\beta_j e_{jj}(t) - \sum_{k=1}^{n} s_k c_{jk}(t) - \sum_{k=1,\ k\neq j}^{m} \beta_k e_{kj}(t) \geqslant \alpha(t) \quad \text{for all } t \in \mathbb{R},\ j = 1, \dots, m,$$

then system (4.1) has a unique globally attractive solution $(u^*(.), v^*(.)) \in \mathcal{B}_+^{n+m}$*.*

Proof. The existence of a solution $(u^*(t), v^*(t))$ follows from Corollary 4.1.
(i) *The uniqueness.* For the contrary, suppose that there are two distinct solutions $(u^1(t), v^1(t))$ and $(u^2(t), v^2(t))$ of system (4.1) defined on $\mathbb{R}$ and satisfying $u_i^l(t) \in [\delta, \Delta]$, $v_j^l(t) \in [\delta, \Delta]$ for all $t \in \mathbb{R}$, $i = 1, \dots, n$, $j = 1, \dots, m$ and $l = 1, 2$, where δ and Δ are positive constants. Let $(u^1(t_0), v^1(t_0)) \neq (u^2(t_0), v^2(t_0))$ for some $t_0 \in \mathbb{R}$. Let $V(t) = \sum_{i=1}^{n} s_i |\ln u_i^1(t) - \ln u_i^2(t)| + \sum_{j=1}^{m} \beta_j |\ln v_j^1(t) - \ln v_j^2(t)|$. Then

$$
\begin{aligned}
D^+V(t) \leqslant & \sum_{i=1}^{n}\Big[\sum_{k=1,\ k\neq i}^{n} s_k a_{ki}(t) + \sum_{k=1}^{m}\beta_i d_{ki}(t) - s_i a_{ii}(t)\Big]|u_i^1(t) - u_i^2(t)| \\
& + \sum_{j=1}^{m}\Big[\sum_{k=1}^{n} s_k c_{kj}(t) + \sum_{k=1,\ k\neq j}^{m}\beta_k e_{kj}(t) - \beta_j e_{jj}(t)\Big]|v_j^1(t) - v_j^2(t)| \\
\leqslant & - \alpha(t)\Big\{\sum_{i=1}^{n}|u_i^1(t) - u_i^2(t)| + \sum_{j=1}^{m}|v_j^1(t) - v_j^2(t)|\Big\} \leqslant -\gamma\alpha(t)V(t),
\end{aligned}
$$

where $\gamma = \min\Big\{\dfrac{\delta}{s_i},\ \dfrac{\delta}{\beta_j} : i = 1, \dots, n,\ j = 1, \dots, m\Big\}$. Thus,

$$
0 < V(t_0) \leqslant V(t)\exp\Big\{-\int_t^{t_0}\gamma\alpha(s)ds\Big\},\ t \leqslant t_0.
$$

Since $V(t)$ is bounded and $\lim\limits_{t\to-\infty}\exp\Big\{-\int_t^{t_0}\gamma\alpha(s)ds\Big\} = 0$, we have $V(t_0) = 0$. This is a contradiction. The uniqueness is proved.

(ii) The global attractivity. Let $(u(t), v(t))$ be a solution of (4.1) with $(u(t_0), v(t_0)) \in \operatorname{int}\mathbb{R}^{n+m}$. By Corollary 4.1, there exist $\delta > 0, \Delta > 0$ and $T \geqslant t_0$ such that $(u(t), v(t)), (u^*(t), v^*(t)) \in [\delta, \Delta]^{n+m}$ for all $t \geqslant T$. Let $V(t) = \sum\limits_{i=1}^{n} s_i|\ln u_i(t) - \ln u_i^*(t)| + \sum\limits_{j=1}^{m}\beta_j|\ln v_j(t) - \ln v_j^*(t)|$. By calculating the upper right derivative of $V(t)$ as given above, we obtain $D^+V(t) \leqslant -\gamma\alpha(t)V(t)$ for $t \geqslant T$, where $\gamma = \min\limits_{i,j}\Big\{\dfrac{\delta}{s_i},\ \dfrac{\delta}{\beta_j}\Big\}$. Thus, $V(t) \leqslant V(T)\exp\Big\{-\int_T^t\gamma\alpha(s)ds\Big\}$ for each $t \geqslant T$. This implies that $\lim\limits_{t\to+\infty}V(t) = 0$, then $\lim\limits_{t\to+\infty}\|(u(t), v(t)) - (u^*(t), v^*(t))\| = 0$. □

THEOREM 4.3. *Let $a_{ik}(t)$, $c_{ik}(t)$, $d_{jk}(t)$, $e_{jk}(t)$, $b_i(t)$ and $r_j(t)$ ($i = 1, \dots, n$, $j = 1, \dots, m$) be almost periodic. Assume that*

$$
\liminf_{T\to+\infty}\frac{1}{T}\int_0^T b_i(s)ds > 0,\ \liminf_{T\to+\infty}\frac{1}{T}\int_0^T a_{ii}(s)ds > 0,\ i = 1, \dots, n, \tag{4.6}
$$

$$
\liminf_{T\to+\infty}\frac{1}{T}\int_0^T e_{jj}(s)ds > 0,\ \liminf_{T\to+\infty}\frac{1}{T}\int_0^T\Big[r_j(s) + \sum_{k=1}^{m} d_{jk}(s)U_k^0(s)\Big]ds > 0, \quad j = 1, \dots, m, \tag{4.7}
$$

$$\text{(4.8)}\quad \liminf_{T\to+\infty}\frac{1}{T}\int_0^T\Big[b_i(s)-\sum_{k=1,\ k\neq i}^{n}a_{ik}(s)U_k^0(s)-\sum_{k=1}^{m}c_{ik}(s)V_k^0(s)\Big]ds>0,$$
$$i=1,\dots,n,$$

$$\text{(4.9)}\quad \liminf_{T\to+\infty}\frac{1}{T}\int_0^T\Big[r_j(s)+\sum_{k=1}^{m}d_{jk}(s)u_k^0(s)-\sum_{k=1,\ k\neq j}^{m}e_{jk}(s)V_k^0(s)\Big]ds>0,$$
$$j=1,\dots,m$$

where $U_i^0(.)$ ($u_i^0(.)$ and $V_j^0(.)$) is the unique almost periodic solution in $\mathcal{B}_+$ of (4.2_i), ((4.4_i) and (4.3_j), respectively). Then (4.1) is permanent and it has least one solution $(u^*(.),v^*(.))\in\mathcal{B}_+^{n+m}$. If, in addition, ($L_7$) holds, then there exists a unique globally attractive almost periodic solution $(u^*(.),v^*(.)\in\mathcal{B}_+^{n+m}$ and its module is contained in that of $F(t,u,v)$, where $F(t,u,v)$ is the right hand side of (4.1). In particular, if $a_{ik}(t)$, $c_{ik}(t)$, $d_{jk}(t)$, $e_{jk}(t)$, $b_i(t)$ and $r_j(t)$ ($i=1,\dots,n,\ j=1,\dots,m$) are Θ-periodic, then also the above solution $(u^*(.),v^*(.))$ is Θ-periodic.

Proof. By Corollary 4.1, system (4.1) is permanent and it has least one solution $(u^*(.),v^*(.))\in\mathcal{B}_+^{n+m}$. We know that for each $F^*\in H(F)$ (the hull of F), there exist $a_{ik}^*\in H(a_{ik})$, $c_{ik}^*\in H(c_{ik})$, $d_{jk}^*\in H(d_{jk})$, $e_{jk}^*\in H(e_{jk})$, $b_i^*\in H(b_i)$ and $r_j^*\in H(r_j)$ ($i=1,\dots,n,\ j=1,\dots,m$) such that $F^*(t,u,v)$ is the right hand side of the following system

$$\text{(4.10)}\quad \begin{aligned}\dot u_i&=u_i\Big[b_i^*(t)-\sum_{k=1}^{n}a_{ik}^*(t)u_k-\sum_{k=1}^{m}c_{ik}^*(t)v_k\Big],\ i=1,\dots,n,\\ \dot v_j&=v_j\Big[r_j^*(t)+\sum_{k=1}^{n}d_{jk}^*(t)u_k-\sum_{k=1}^{m}e_{jk}^*(t)v_k\Big],\ j=1,\dots,m.\end{aligned}$$

For $i=1,\dots,n$ and $j=1,\dots,m$, let us consider

$$\text{(4.11}_i\text{)}\quad \dot u_i=u_i[b_i^*(t)-a_{ii}^*(t)u_i],$$

$$\text{(4.12}_j\text{)}\quad \dot v_j=v_j\Big[r_j^*(t)+\sum_{k=1}^{m}d_{jk}^*(t)U_k^{*0}(t)-e_{jj}^*(t)v_j\Big],$$

$$\text{(4.13}_i\text{)}\quad \dot u_i=u_i\Big[b_i^*(t)-\sum_{k=1,\ k\neq i}^{n}a_{ik}^*(t)U_k^{*0}(t)-\sum_{k=1}^{m}c_{ik}^*(t)V_k^{*0}(t)-a_{ii}^*(t)u_i\Big],$$

$$\text{(4.14}_j\text{)}\quad \dot v_j=v_j\Big[r_j^*(t)+\sum_{k=1}^{m}d_{jk}^*(t)u_k^{*0}(t)-\sum_{k=1,\ k\neq j}^{m}e_{jk}^*(t)V_k^{*0}(t)-e_{jj}^*(t)v_j\Big].$$

By Lemma 2.7, each of equations (4.11_i), (4.12_j), (4.13_i), (4.14_j) has a unique almost periodic solution $U_i^{*0}(.)$, $V_j^{*0}(.)$, $u_i^{*0}(.)$ and $v_j^{*0}(.)$ in $\mathcal{B}_+$, respectively. Let $\{\tau_k\}_{k=1}^{\infty}$ be a sequence of numbers such that $b_{i\tau_k} \to b_i^*$, $a_{ii\tau_k} \to a_{ii}^*$ as $k \to \infty$ uniformly on $\mathbb{R}$. Without loss of generality, we may assume that $U_{i\tau_k}^0 \to \bar{U}_i^0$ as $k \to \infty$ uniformly on $\mathbb{R}$. It is easy to see that $\bar{U}_i^0$ is a solution of equation (4.11_i) and thus $U_i^{*0}(.) \equiv \bar{U}_i^0(.)$. This implies that $\sup\limits_{t\in\mathbb{R}} U_i^{*0}(t) = \sup\limits_{t\in\mathbb{R}} U_i^0(t)$. Similarly, $\sup\limits_{t\in\mathbb{R}} V_j^{*0}(t) = \sup\limits_{t\in\mathbb{R}} V_j^0(t)$, $\inf\limits_{t\in\mathbb{R}} u_i^{*0}(t) = \inf\limits_{t\in\mathbb{R}} u_i^0(t)$, $\inf\limits_{t\in\mathbb{R}} v_j^{*0}(t) = \inf\limits_{t\in\mathbb{R}} v_j^0(t)$. Clearly that $\sup\limits_{(t,u,v)\in\mathbb{R}\times S} |F_k^*(t,u,v)| = \sup\limits_{(t,u,v)\in\mathbb{R}\times S} |F_k(t,u,v)|$ for any compact set $S \subset \mathbb{R}^{n+m}$. Let

$$\begin{aligned}
\delta &= \inf\{u_i^0(t), v_j^0(t) \colon i = 1,\dots,n,\ j = 1,\dots,m,\ t \in \mathbb{R}\},\\
\Delta &= \sup\{U_i^0(t), V_j^0(t) \colon i = 1,\dots,n,\ j = 1,\dots,m,\ t \in \mathbb{R}\},\\
L &= \max_{k=1,\dots,n+m} \Big\{ \sup_{(t,u,v)\in\mathbb{R}\times[\delta,\Delta]^{n+m}} |F_k^*(t,u,v)| \Big\}.
\end{aligned}$$

By the same argument as given in the proof of Theorem 3.1, we know that system (4.10) has at least one solution $(\bar{u}(t), \bar{v}(t))$ in $\mathcal{M}_1^*$ where

$$\begin{aligned}
\mathcal{M}_1^* = \big\{(u(.), v(.)) : {}& (u^{*0}(t), v^{*0}(t)) \leqslant (u(t), v(t)) \leqslant (U^{*0}(t), V^{*0}(t)),\\
& |u_i(t) - u_i(\bar{t})| \leqslant L|t - \bar{t}|,\ i = 1,\dots,n,\\
& |v_j(t) - v_j(\bar{t})| \leqslant L|t - \bar{t}|,\ j = 1,\dots,m,\ t, \bar{t} \in \mathbb{R}\big\}.
\end{aligned}$$

It is easy to see that system (4.10) satisfies all conditions in Theorem 4.2. Thus, for each $F^* \in H(F)$, system (4.10) has a unique solution $(\bar{u}(t), \bar{v}(t))$ with $(\bar{u}(t), \bar{v}(t)) \in [\delta, \Delta]^{n+m}$ for all $t \in \mathbb{R}$. Since δ and Δ do not depend on the choice of $F^* \in H(F)$, from Lemma 2.6 and Theorem 4.2 it follows that there exists a unique globally attractive almost periodic solution $(u^*(.), v^*(.)) \in \mathcal{B}_+^{n+m}$ of system (4.1). Moreover, the module of $(u^*(t), v^*(t))$ is contained in that of $F(t,u,v)$. If F is Θ-periodic in t, then $(u^*(.), v^*(.))$ and $(u_\Theta^*(.), v_\Theta^*(.))$ are two solutions in $\mathcal{B}_+^{n+m}$ of (4.1). By the uniqueness, $(u^*(.), v^*(.)) = (u_\Theta^*(.), v_\Theta^*(.))$. The theorem is proved. □

Remark. In [**7**], the authors considered system (4.1) with $b_i(t)$, $-r_j(t)$, $a_{ik}(t)$ $(i \neq k)$, $e_{jl}(t)$ $(j \neq l)$, $c_{il}(t)$ and $d_{jk}(t)$ nonnegative almost periodic; $a_{ii}(t)$ and $e_{jj}(t)$ are almost periodic and bounded from above and from below by positive constants. If $f : \mathbb{R} \to \mathbb{R}$ is almost periodic, we set $f^h = \inf\limits_{t\in\mathbb{R}} f(t)$ and $f^H = \sup\limits_{t\in\mathbb{R}} f(t)$. Moreover, we set

$$p_i = \frac{b_i^H}{a_{ii}^h},\quad q_j = \frac{1}{e_{jj}^h}\Big(\sum_{k=1}^n d_{jk}^H p_k + r_j^H\Big),\quad \alpha_i = \frac{1}{a_{ii}^H}\Big(b_i^h - \sum_{k=1,k\neq i}^n a_{ik}^H p_k - \sum_{k=1}^m c_{ik}^H q_k\Big),$$

$$\beta_j = \frac{1}{e_{jj}^H}\Big(r_j^h + \sum_{k=1}^{n} d_{jk}^h \alpha_k - \sum_{k=1,k\neq j}^{m} c_{jk}^H q_k\Big),\ i = 1,\dots,n,\ j = 1,\dots,m.$$

In [**7**] it was shown that: *If*

$$\alpha_i > 0,\ \beta_j > 0,\ q_j > 0 \tag{4.15}$$

and (L_7) *hold, then system (4.1) has a unique globally attractive almost periodic solution* $(u^*(.), v^*(.)) \in \mathcal{B}_+^{n+m}$ *and its module is contained in that of* $F(t,u,v)$, *where* $F(t,u,v)$ *is the right hand side of (4.1).*

It is easy to see that $\sup_{t\in\mathbb{R}} U_i^0(t) \leqslant p_i$ $(i = 1,\dots,n)$ and $\sup_{t\in\mathbb{R}} V_j^0(t) \leqslant q_j$ $(j = 1,\dots,m)$. Thus condition (4.15) implies conditions (4.6), (4.7), (4.8) and (4.9). The following example shows that Theorem 4.3 generalizes and improves the above result in [**7**].

Example. Consider the following system

$$\begin{aligned} \dot{u} &= u[(0.5-0.5(\cos t+\cos\sqrt{2}t))-(1.1-0.5(\cos t+\cos\sqrt{2}t))u-0.04v],\\ \dot{v} &= v[\sin t+\sin\sqrt{3}t+u-v]. \end{aligned} \tag{4.16}$$

By Lemma 2.7, the equation $\dot{u} = u[0.5-0.5(\cos t+\cos\sqrt{2}t)-(1.1-0.5(\cos t+\cos\sqrt{2}t))u]$ has a unique almost periodic solution $U^0(.) \in \mathcal{B}_+$. It is easy to see that

$$\sup_{t\in\mathbb{R}} U^0(t) \leqslant \sup_{t\in\mathbb{R}} \frac{0.5-0.5(\cos t+\cos\sqrt{2}t)}{1.1-0.5(\cos t+\cos\sqrt{2}t)} \leqslant \frac{1.5}{2.1}.$$

By Lemma 2.7, the equation $\dot{v} = v[\sin t+\sin\sqrt{3}t+U^0(t)-v]$ has a unique almost periodic solution $V^0(.) \in \mathcal{B}_+$. Since $\lim_{T\to+\infty}\frac{1}{T}\int_0^T V^0(t)dt = \lim_{T\to+\infty}\frac{1}{T}\int_0^T [\sin t + \sin\sqrt{3}t + U^0(t)]dt \leqslant \frac{1.5}{2.1}$, we have $\lim_{T\to+\infty}\frac{1}{T}\int_0^T [0.5-0.5(\cos t+\cos\sqrt{2}t) - 0.04V^0(t)]dt > 0$. It follows that the equation

$$\dot{u} = u[(0.5-0.5(\cos t+\cos\sqrt{2}t)) - 0.04V^0(t) - (1.1-0.5(\cos t+\cos\sqrt{2}t))u]$$

has a unique almost periodic solution $u^0(.) \in \mathcal{B}_+$. Now, it is easy to verify that system (4.1) satisfies all conditions (4.6)–(4.9). Moreover, condition (L_7) holds for $s = 0.5$, $\beta = 0.04$. Therefore, by Theorem 4.3, system (4.16) has a unique globally attractive almost periodic solution $\big(u^*(.), v^*(.)\big) \in \mathcal{B}_+^2$, whereas system (4.16) does not satisfy (4.15).

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