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## LOCALLY ORDERED TOPOLOGICAL SPACES

**A b s t r a c t.** While topology given by a linear order has been extensively studied, this cannot be said about the case when the order is given only locally. The aim of this paper is to fill this gap. We consider relation between local orderability and separation axioms and give characterisation of those regularly locally ordered spaces which are connected, locally connected or Lindelöf. We prove that local orderability is hereditary on open, connected or compact subsets. A collection of interesting examples is also offered.

### 1. Introduction

Formal definition of an ordered set appeared in 1880 due to C. S. Pierce but the idea was somehow present in mathematics and philosophy long before. While mentioning early research on ordered sets names of Dedekind and Cantor cannot be omitted. Some historical notes can be found in [6].

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The concept of order topology appeared probably at the same time as abstract topology itself. On the one hand, it is a very classical notion today, on the other, there were studied various different connections between topology and order on a set (see e.g. [4] or [6]). Several notable results concerning linearly ordered spaces and their subspaces were listed in [7]. For more recent works the reader is referred to [1].

The natural notion of a locally ordered topological space, which we consider in this article, seem to had appeared only once, in not easily accessible dissertation by Horst Herrlich [3]. Several of his results are similar to those presented in this article; however, we omit the notion of end-finite space (i.e. space with at most two non-cut points for each connected component) and put some attention to the case when a locally ordered space is not orderable. For example the heredity of local orderability on arbitrary connected or compact subsets was not discussed by Herrlich.

The aim of this paper is to present general results concerning locally ordered spaces and the most classical topological notions. Our survey starts with basic definitions and observations concerning separation axioms and heredity of local orderability. Then we pass to properties of connected and locally connected spaces and prove characterisation of all connected  $T_3$  locally ordered spaces (Theorem 3.6). This leads also to description of both locally connected (Theorem 3.11) and Lindelöf (Theorem 4.2) among such spaces. Provided characterisation is valid also for arbitrary connected or compact subsets of a locally ordered  $T_3$  space.

All notable examples of locally ordered spaces are presented in the separate section. Some of them are well known to topologists but possibly not for their local orderability.

## 1.1 Notation and terminology

We are not going to denote topological spaces formally as pairs  $(X, \tau)$ . Since we never refer to two different topologies on one set at a time, the risk of ambiguity is minimal.

By neighbourhood of a point we mean an open set containing this point.

As we will see at the very beginning, all spaces of concern would be  $T_1$  hence we are not going to distinguish e.g. between “ $T_4$  space” and “normal space”. By *Urysohn space* ( $T_{2\frac{1}{2}}$ ) we mean space in which every two distinct points have neighbourhoods with disjoint closures. We call a space *completely Hausdorff* if its any two distinct points can be separated by a real-valued function. By *semiregular space* we mean Hausdorff space which has a basis consisting of sets being interiors of their closures.

For a set  $X$ , we call the set  $\Delta_X := \{(x, x) : x \in X\}$  the *diagonal of  $X$* .

When considering ordered sets we refer to strict (irreflexive) linear order relations (cf. [2]). In most cases we do not denote the ordering relations explicitly, similarly like the topology on a set. We write “ $K$  is an open interval with respect to the order on  $U$ ” (or simply “ $K$  is an open interval in  $U$ ”) as long as it is a sufficient clarification. The natural notation “ $(a, b)_U$ ” for an (open) interval in  $U$  is also used. Symbols like “ $(\leftarrow, b]$ ” and “ $(a, \rightarrow)$ ” denote unbounded intervals. Note that when we write “open interval” we do not assume its boundedness.

By a closed interval we mean an interval including both its least and its greatest element – not any interval which is a closed set. For open intervals there is no such ambiguity.

A subset of a linearly ordered set is called *convex* if for its every two points it contains the interval spanned by them. The term *endpoint* stands for the supremum or the infimum of a convex set (they may not exist).

For two subsets  $A$  and  $B$  of a linearly ordered set by “ $A < B$ ” we mean that for every  $a \in A$  and  $b \in B$  the inequality  $a < b$  is satisfied.

## 2. Basic definitions and properties

### 2.1 Order topology

First let us recall basic facts about classical order topology. They belong to the folklore and are mostly mentioned as exercises (see e.g. [2], [11]).

**Definition 2.1.** Given a linearly ordered set  $(X, <)$ , by *order topology* (called also *open interval topology*) we mean a topology defined by the basis consisting of all open intervals in  $(X, <)$ . Space  $X$  with an order topology is called *linearly ordered (topological) space* or *orderable space* (if the order on  $X$  is not fixed). We call a linear order on  $X$  *compatible with the topology* if the associated order topology equals the topology on  $X$ .

Unless it is stated otherwise, whenever we mention an order on a topological space (or subspace) we mean an order compatible with the topology.

A *suborderable* (or *generalised ordered*) space is a topological space homeomorphic to a subspace of a linearly ordered space.

Every linearly ordered (orderable) space is hereditarily normal ( $T_5$ ) – even hereditarily collectionwise normal (see [10]).

Every connected subset of a linearly ordered topological space has to be convex. Closure and interior of a convex set are convex. Maximal convex subset in a closed set is closed and maximal convex subset in an open set is open.

A linear ordering  $<$  on  $X$  is called *continuous* if it is dense and every convex set is an interval (possibly unbounded)<sup>1</sup>. This is equivalent to the connectedness of associated order topology (cf. [2, Problem 5.3.2]). A subset of a connected linearly ordered space is connected if and only if it is an interval. If a linearly ordered space is not connected then it is a disjoint union of two non-empty convex and open sets. In order theory such separation is called a *cut* or, more specifically, either a *gap* or a *jump*.

On a connected orderable space containing at least two points there are precisely two linear orders (each one is the reverse of the other) compatible with the topology. Although the fact is long-known (e.g. [3, II.]), we provide an “exceptionally topological” proof in the Appendix (Theorem 6.1).

Compactness of order topology is equivalent to the existence of supremum and infimum for any subset (cf. [2, Problem 3.12.3 (a)]). For a connected space it is enough to check whether it has both the smallest and the greatest element. Every connected linearly ordered space is automatically locally connected and locally compact. Endpoints of a connected linearly ordered space can be characterised topologically (i.e. they are precisely non-cutpoints), hence we can consider endpoints of a connected orderable space (regardless of the linear order).

On arbitrary compact subset of a linearly ordered space the subspace topology and the order topology given by the linear order inherited from the original space coincide ([3, I. Satz 13c]). In particular orderability is hereditary on compact subspaces.

## 2.2 Locally order topology

Now we can pass to main definitions of this article.

**Definition 2.2.** Let  $X$  be a topological space. We say  $X$  is *locally ordered* (or *has locally order topology*) if each point in  $X$  has an orderable neighbourhood. An open cover of  $X$  consisting of linearly ordered sets with fixed linear orders will be called an *atlas of orders*.

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<sup>1</sup>The standard order on rational numbers is dense, but the convex set  $\{x \in \mathbb{Q} : x^2 < 2\}$  is not an interval in  $\mathbb{Q}$ . Every convex subset of  $\mathbb{N}$  is an interval but the order is not dense.

We say  $X$  is *regularly locally ordered* if each point in  $X$  has a neighbourhood whose closure is orderable. Then a *regular atlas of orders* is an open cover of  $X$  together with fixed linear orders on the closures.

Note that at the beginning we do not assume that considered spaces satisfy any separation axiom.

Let us start from

**Proposition 2.3.**

1. *Every space with order topology is regularly locally ordered.*
2. *Every one dimensional topological manifold is regularly locally ordered.*

In particular a circle is locally ordered but not orderable at the same time and cannot be embedded in a linearly ordered space. Later we are introducing a whole class of spaces sharing those properties.

The following theorem is also a simple observation.

**Proposition 2.4.** *Every open subspace of locally ordered space is itself locally ordered.*

**Proof.** Let us observe that every interval in a space with order topology is itself a linearly ordered topological space (order topology and subspace topology coincide). Since open subspace for each of its points contains some interval with respect to the order on a neighbourhood, it satisfies the definition of local orderability. □

This behaviour is different from the case of linearly ordered spaces, e.g.  $[0, 1] \cup \{2\}$  with Euclidean topology is a linearly ordered space containing a not orderable open subset  $(0, 1) \cup \{2\}$ .

The property of local orderability is not hereditary in general.

**Example 2.5.** The set  $(-1, 0] \cup \bigcup_{n=1}^{\infty} \left( \left\{ \frac{1}{3n} \right\} \cup \left( \frac{1}{3n-1}, \frac{1}{3n-2} \right) \right)$  with the topology induced from  $\mathbb{R}$  is not locally ordered.

**Proof.** Assume 0 has some linearly ordered neighbourhood  $U$ . Clearly the connected component at 0 (an interval of the form  $(s, 0]$ ) has to be some interval closed at 0. Without loss of generality we can assume that it consists of elements not greater than 0 with respect to the order on  $U$ . The rest of  $U$  has countably many connected components, namely singletons and sets homeomorphic to  $(0, 1)$ . Both isolated points and intervals converge to 0. There exist such singleton and interval that there is no element between

them, because otherwise there will be an infinite set of intervals or singletons between some two leading to contradiction with their convergence to 0. Hence the singleton must be in the closure of open interval while it is not the case.  $\square$

**Remark 2.6.** The example was known to Herrlich ([3]). A different idea is presented as a part of the Example 5.9 below.

One can notice that the space introduced in the Example 2.5 is a closed subset of some orderable subspace of  $\mathbb{R}$ . It will be shown later (Proposition 4.10) that local orderability is hereditary on compact subspaces.

At this point we know that there is no inclusion between the classes of locally ordered spaces and generalised ordered spaces (a.k.a. suborderable spaces).

The following facts deal with separation axioms and also explain why stronger version of local orderability property is called *regular* local orderability.

**Lemma 2.7.** *Every locally ordered space is  $T_1$ .*

**Proof.** Given two distinct points  $x, y$  of a locally ordered space, either  $y$  does not belong to an ordered neighbourhood of  $x$  from the atlas of orders or we can use the fact that an ordered neighbourhood of  $x$  is  $T_1$  itself.  $\square$

**Theorem 2.8.** *For a locally ordered space  $X$  the following conditions are equivalent:*

- (a)  $X$  is regularly locally ordered
- (b)  $X$  is regular ( $T_3$ )
- (c)  $X$  is Tychonoff ( $T_{3\frac{1}{2}}$ ).

**Proof.** The implication (c) $\Rightarrow$ (b) is simple and well known.

(a) $\Rightarrow$ (c) Fix  $A$ , a closed subset of a regularly locally ordered space  $X$ , and a point  $x \in X \setminus A$ . Let  $U$  be a neighbourhood of  $x$  such that  $\bar{U}$  is orderable. Linearly ordered spaces are  $T_3$  (even  $T_5$ ), so we can find disjoint open sets  $V$  and  $W$  in  $\bar{U}$  such that  $x \in V$  and  $A \cap U \subseteq W$ . Note that  $\bar{U} \setminus W$  is closed in  $X$ .

The set  $V \cap U$  is then a neighbourhood of  $x$  in  $X$  disjoint from  $X \setminus (\bar{U} \setminus W)$ , a neighbourhood of  $A$ . We can pick any function on  $\bar{U}$  separating  $x$  and

$\bar{U} \setminus (V \cap U)$  (since linearly ordered spaces are Tychonoff) and extend it by a constant on  $X \setminus \bar{U}$ .

(b) $\Rightarrow$ (a) Fix  $x$ , a point in a locally ordered  $T_3$  space. It has an orderable neighbourhood  $U$ . Pick an open set  $V$  such that  $x \in V \subseteq \bar{V} \subseteq U$ . If we find then an open interval inside  $V$  containing  $x$ , its closure will be contained in  $U$ , and hence will be an interval which is orderable.  $\square$

**Corollary 2.9.** *On a regularly locally ordered space there exists a regular atlas of orders  $\{(U_i, <_i)\}_{i \in I}$  such that for every  $i \in I$  the set  $U_i$  is an open interval in the linearly ordered space  $(\bar{U}_i, <_i)$ .*

**Proof.** Construction of such regular atlas of orders is presented in the proof of the previous lemma (b $\Rightarrow$ a).  $\square$

**Corollary 2.10.** *Every locally ordered  $T_3$  space is completely Hausdorff.*

On the contrary, a completely Hausdorff locally ordered space may not be  $T_3$ .

For locally ordered spaces there are no other implications between low separation axioms (namely:  $T_1, T_2, T_{2\frac{1}{2}}$ , semiregularity and complete Hausdorff) than those valid for topological spaces in general. The table below lists the sufficient counterexamples.

Example	5.3	5.4	5.5	5.11	5.6
Hausdorff	0	1	1	1	1
Urysohn	0	0	1	1	1
completely Hausdorff	0	0	0	1	1
semiregular	0	1	1	0	1
$T_3$	0	0	0	0	0

All mentioned examples are second-countable connected locally ordered topological spaces.

Basing on the Example 5.11 one can modify spaces from Examples 5.4 and 5.5 to be not semiregular.

Since both local orderability and  $T_3$  are hereditary on open subspaces, so is regular local orderability. Further, we will show that regular local orderability is also hereditary on connected (Proposition 3.12) and compact (Proposition 4.10) subsets.

**Remark 2.11** (On higher separation axioms).

1. A regularly locally ordered space need not to be normal (see Examples 5.10 and 5.12).
2. The long line (see e.g. [9]) is an example proving that a  $T_5$  locally ordered space need not to be  $T_6$  (not every closed set is  $G_\delta$ ).
3. For a linearly ordered space,  $T_6$  implies the first axiom of countability but the opposite implication is not true (e.g. the long line). It follows that a locally ordered  $T_6$  space is first-countable.
4. Second-countable locally ordered space may not be even Hausdorff (Example 5.3). First-countable regularly locally ordered space may not be normal (Example 5.10). For regularly locally ordered spaces the second axiom of countability is strong assumption implying the decomposition described in Corollary 4.3.
5. It is also known that linearly ordered spaces which are  $T_6$  may not be metrizable. As an example one can take the set  $[0, 1] \times [0, 1]$  with the lexicographic order (see [9]).

Lutzer ([5]) proved that a linearly ordered space  $X$  is metrizable if and only if the diagonal  $\Delta_X$  is a  $G_\delta$  subset of the product space  $X \times X$ . The Example 5.10 shows that such condition is not sufficient for regularly locally ordered spaces. It obviously implies local metrizability, hence, keeping in mind well known metrization theorem by Smirnov ([8], [2, 5.4.A]), we can formulate the following characterisation of metrizability for locally ordered spaces.

**Theorem 2.12.** *A locally ordered space is metrizable if and only if it is paracompact Hausdorff and has a  $G_\delta$  diagonal.*

### 3. Connectedness

For linearly ordered topological spaces connectedness implies local connectedness and local compactness. This is not true in general for locally ordered spaces (see Examples 5.6 and 5.11).

**Lemma 3.1.**

- a) *A connected regularly locally ordered space is locally connected.*

b) *Locally connected Hausdorff locally ordered space is locally compact (and hence regular).*

**Proof.** a) Assume that space  $X$  is locally ordered,  $T_3$  and not locally connected. We will prove that then  $X$  is not connected.

Consider  $x \in X$ , a point without a connected, ordered neighbourhood (if there is no such point, the space is locally connected, since connectedness of an ordered space implies its local connectedness). Let  $W$  be a neighbourhood of  $x$  such that it is an interval with respect to the fixed order on  $\overline{W}$  (Corollary 2.9).

Observe that it is impossible for both of the sets  $(\leftarrow, x)_W$  and  $(x, \rightarrow)_W$  to be connected simultaneously. Furthermore, after reversing the order if necessary, we can assume that for no  $y \in (x, \rightarrow)_W$  the interval  $(x, y)_W$  is connected. We pick separation of  $(x, \rightarrow)_W$  into open convex sets  $W_1$  and  $W_2$  such that  $W_1 < W_2$ . Then for  $y_1 \in W_1$  we can find a separation of  $(x, y_1)_W$  into open convex sets  $V_1$  and  $V_2$ , where  $V_1 < V_2$ . Let  $V := W_1 \setminus V_1$ . Observe that since each of the above separations of open intervals can be naturally extended to a separation of a closed interval, the set  $V$  is simultaneously closed and open in  $W$ .

Picking any  $y_2 \in W_2$  we know that  $V \subseteq [x, y_2]_{\overline{W}}$  and hence the closure of  $V$  in  $X$  equals  $V$ , i.e. its closure in  $\overline{W}$ . Hence the closed sets  $V$  and  $X \setminus V$  separate space  $X$ .

b) Denote given locally connected locally ordered Hausdorff space by  $X$ . Fix point  $x$  and its linearly ordered neighbourhood  $U$ . It follows from the local connectedness that inside  $U$  there exists a connected neighbourhood  $C$  of  $x$ . Such  $C$  is convex and then orderable. Connected and orderable space is locally compact, hence there exists a neighbourhood  $V$  of  $x$  with compact closure in  $C$ . Such closure is closed in the Hausdorff space  $X$  and hence  $\overline{V}$  is compact. □

Hausdorff axiom is important, since  $\mathbb{R}$  with doubled origin (Example 5.3) is connected and locally connected but definitely not  $T_3$ .

Now let us define an important class of locally ordered spaces, a generalisation of the circle.

**Definition 3.2.** A topological space obtained from a compact and connected linearly ordered space (containing at least two points) by identification of the smallest and the greatest element is called a *loop-ordered topological space*.

Assuming connectedness in the definition of loop-ordered space is important, because otherwise after the identification of the endpoints we would still obtain an orderable space.

Proofs of the following simple properties of loop-ordered spaces are left to the reader.

**Proposition 3.3.** *Let  $X$  be a loop ordered space. Then*

- a)  $X$  is compact and connected regularly locally ordered space,
- b)  $X$  is hereditarily normal ( $T_5$ ),
- c)  $X$  cannot be homeomorphically embedded in a linearly ordered space,
- d) for any  $x \in X$  the subspace  $X \setminus \{x\}$  is connected, orderable and not compact.

**Remark 3.4.** Due to topological characterisations of the unit interval, every metrizable loop-ordered space is homeomorphic to the unit circle in  $\mathbb{R}^2$ . It can be also deduced from [2, 6.3.2c.].

The following simple fact would be useful in the future description of locally ordered spaces.

**Lemma 3.5.** *In a locally ordered space every loop-ordered subspace is open.*

**Proof.** Fix a point  $x_0$  in a loop ordered subspace  $L$ . If  $U$  is an ordered neighbourhood of  $x_0$ , then some connected neighbourhood of  $x_0$  in  $L$  has to be contained in  $U$ . A connected subset of  $U$  has to be convex, and  $x_0$  is not its endpoint being a cutpoint. Hence there exists an open interval containing  $x_0$  and enclosed in  $L$ . Since we can find such interval for any  $x_0$  in  $L$ , the loop-ordered subset is open.  $\square$

Now we can formulate the main result of the paper, namely classification theorem for connected regularly locally ordered spaces.

**Theorem 3.6.** *If  $X$  is a connected regularly locally ordered topological space, then  $X$  is either an orderable space or a loop-ordered space.*

Before we prove it let us start with the following simpler case.

**Lemma 3.7.** *If a connected Hausdorff space can be covered by two open connected and orderable sets then it is either an orderable space or a loop-ordered space.*

*Moreover, the space is orderable if and only if the intersection of the aforementioned open sets is connected.*

**Proof.** Consider  $U$  and  $V$ , two open connected linearly ordered subsets of  $X = U \cup V$ . We can skip the trivial case when  $X$  consists of less than two points.

The intersection  $U \cap V$  has to be a disjoint union of open (possibly unbounded) intervals in  $U$ , namely its connected components.

For the use of this proof we say a subset  $A$  of a linearly ordered space is bounded from below (above) if there exists a strict lower (upper) bound of this set (an element strictly smaller/greater than any element of  $A$ ). For example the interval  $[0, 1]$  is, in this sense, not bounded in itself, while it is bounded in  $\mathbb{R}$ .

Assume there exists  $W$ , a connected component of  $U \cap V$  bounded from both sides (with respect to the order on  $U$ ). Then its closure in  $U$  (closed interval) is compact, hence closed in  $X$  and therefore  $\overline{W} \cap V \subseteq U$ . It follows from closedness of  $W$  in  $U \cap V$  that  $W = U \cap V \cap \overline{W}$  and hence is closed in  $V$ . Since it is simultaneously open,  $V = W \subseteq U$  and we are done.

Assuming that neither  $U \subseteq V$  nor  $V \subseteq U$  leads then to conclusion that  $U \cap V$  consists of unbounded (from one side) intervals (with respect to the order on  $U$  as well as with respect to the order on  $V$ , since the reasoning is fully symmetric). If there is only one such interval, then we use the fact that there are only two possible orders compatible with a connected order topology so, by reversing order on  $V$  if necessary, we will obtain equality of orders on the intersection. If  $U \setminus V < U \cap V$  (resp.  $U \setminus V > U \cap V$ ) we declare  $x < y$  (resp.  $x > y$ ) for every  $x \in U \setminus V$  and  $y \in V \setminus U$ . Then we get one linear ordering on  $U \cup V$ , extending the one on  $U$ , and compatible with the topology for it agrees locally with orders on  $U$  and  $V$ . This shows the sufficiency of the condition from the “moreover” part.

Now assume that  $U \cap V$  consists of two unbounded intervals. Pick any  $x \in U \setminus V$ . Then  $U = (\leftarrow, x]_U \cup [x, \rightarrow)_U$  and both intervals have connected intersection with  $V$ . Hence, by “splitting the point  $x$ ”<sup>\*</sup> into  $x_R$  and  $x_L$ , we can apply the previous case twice to obtain that  $[x_R, \rightarrow) \cup V \cup (\leftarrow, x_L]$  is a compact orderable space (connected, with both endpoints). Identifying  $x_R$  and  $x_L$  leads to a loop-ordered space homeomorphic to  $X$ . The necessity of the condition in the “moreover” part is proven.

<sup>\*</sup> Note that there are subtle details in the operation of “point splitting”. We define  $[x_R, \rightarrow)$  and  $(\leftarrow, x_L]$  as linearly ordered spaces and observe that the points different from  $x_R$  and  $x_L$  have the same basic neighbourhoods as in the original topology on  $X$ . Then  $(x_R, \rightarrow) \cup V \cup (\leftarrow, x_L)$  equals  $X \setminus \{x\}$  (together with topology). Moreover, the identification of  $x_R$  and  $x_L$  leads to a point with “the same” basic neighbourhoods as the point  $x$ . □

**Proof of Theorem 3.6.** Consider a regular atlas  $\mathcal{U}$  of orders on  $X$  consisting of connected sets. Without loss of generality we can assume that every set from the atlas has at least two points. Otherwise our space trivially has an order topology.

Assume there exists a point  $x_0 \in X$  being an endpoint of its connected orderable neighbourhood  $U_0$ . We may and do assume that it is the smallest element in the order on  $U_0$ . Since  $X$  is connected, for any  $y \in X$  there exists a finite sequence of sets  $U_1, \dots, U_n$  from the atlas  $\mathcal{U}$  such that  $y \in U_n$  and  $U_{j-1} \cap U_j \neq \emptyset$  for  $j = 1, \dots, n$ . We can inductively apply Lemma 3.7 to sets  $\bigcup_{i=0}^{j-1} U_i$  and  $U_j$  to obtain that either the union  $\bigcup_{i=0}^j U_i$  is orderable, or it is a loop-ordered space. Intervals containing  $x_0$  have one-point boundary and hence  $x_0$  cannot be contained in a loop-ordered space, so we can proceed for every  $j = 1, \dots, n$ , obtaining at each step an open orderable subspace of  $X$ .

We proved that every point  $y \in X$  belongs to some open and connected orderable set  $U_y$  containing  $x_0$ . By reversing orders if necessary we can obtain that order on each such connected set agrees with the order on  $U_0$ . They form an open cover  $\mathcal{V} := \{V_y\}_{y \in X}$ . Since  $x_0$  is clearly an endpoint of every set from  $\mathcal{V}$ , it is also an endpoint of intersection of any  $V_y, V_{y'} \in \mathcal{V}$ . We know that  $x_0$  does not belong to a loop-ordered set, hence the intersection  $V_y \cap V_{y'}$  is connected (Lemma 3.7). If  $V_y \cap V_{y'}$  has a supremum ( $x_0$  is the minimum) in both  $V_y$  and  $V_{y'}$ , then those suprema coincide (they have no disjoint neighbourhoods) and hence belong to the set. It means one of the sets  $V_y$  and  $V_{y'}$  is contained in the other and the orders clearly agree. The same conclusion follows the lack of a supremum. Hence we can consider an order on  $X$  being the union of all orders on sets in  $\mathcal{V}$  and such order is compatible with the topology on  $X$ , since it clearly agrees locally.

Now assume that no point in  $X$  is an endpoint of its connected orderable neighbourhood. We will consider two cases.

1. There is a point  $x_0 \in X$  such that  $X \setminus \{x_0\}$  is connected.  $x_0$  is not an endpoint of its ordered neighbourhood  $U_0$ , so we can pick points  $a$  and  $b$  from different components of  $U_0 \setminus \{x_0\}$ . Since  $X \setminus \{x_0\}$  is connected, we can join  $a$  and  $b$  by a sequence of connected orderable neighbourhoods inside  $X \setminus \{x_0\}$ . Their union is not the whole space, so it cannot be a loop-ordered space. We obtain some open connected and orderable set  $V$  containing  $a$  and  $b$ . Then  $V \cup U_0$  has to be a loop-ordered space, for  $U_0 \cap V$  has at least two connected components (Lemma 3.7). Connectedness implies that loop-ordered subset has to be the whole space  $X$ .

2.  $X \setminus \{x\}$  is not connected for any  $x \in X$ . Fix  $x_0 \in X$ . Since any neighbourhood of  $x_0$  splits into at most two components,  $X \setminus \{x_0\}$  also has

two components, let say  $X_1$  and  $X_2$ . Note that both  $X_1 \cup \{x_0\}$  and  $X_2 \cup \{x_0\}$  are connected and regularly locally ordered and have a point (namely  $x_0$ ) being an endpoint of its connected orderable neighbourhood. Hence both those spaces are orderable and they glue together at  $\{x_0\}$  to the orderable space  $X$ .  $\square$

Classification leads to the following simple corollaries.

**Corollary 3.8.** *In a connected regularly locally ordered space there exists at most one pair of distinct points such that their removal does not disconnect the space.*

Note that in general the set of points not separating a connected locally ordered space may be very big. Example 5.7 shows that in a separable space it can be of cardinality continuum.

**Corollary 3.9.** *Every connected but not compact regularly locally ordered space is orderable.*

Several theorems, using notion of end-finiteness (“randendlich”), proven by Herrlich in [3] can be easily derived from classifications presented here, since loop-ordered spaces are certainly not end-finite. Below we present one example.

**Corollary 3.10** (Theorem 2 from Chapter IV in [3]). *A connected locally ordered space is orderable if and only if it is  $T_3$  and end-finite.*

The description of connected regularly locally ordered spaces leads to the following result concerning locally connected spaces.

**Theorem 3.11.** *Every locally connected locally ordered Hausdorff space is a disjoint union of some number of loop-ordered spaces and connected linearly ordered spaces. It is then hereditarily normal ( $T_5$ ) and locally compact.*

**Proof.** The first assertion comes straightforward from the decomposition into connected components which are connected regularly locally ordered spaces (see Lemma 3.1). They are also  $T_5$ , so is their disjoint union.  $\square$

There is one more fact about locally ordered spaces and connectedness.

**Proposition 3.12.** *A connected subset of a regularly locally ordered space is a regularly locally ordered space.*

**Proof.** Fix connected set  $C$ , point  $x \in C$ , its neighbourhood  $U$  with linearly ordered closure and assume  $C \setminus \{x\} \neq \emptyset$  (singletons are obviously regularly locally ordered). We claim that there exists a neighbourhood of  $x$  in  $C$  which is an interval in  $U$ .

First observe that there cannot be an interval around  $x$  not intersecting  $C$ . Otherwise  $x$  would be an isolated point in  $C$ .

We can approach  $x$  by a net in  $C \cap U$ . Without loss of generality we can assume that it is a decreasing net (consisting of elements greater than  $x$  in  $U$ ). There cannot be a decreasing net of points in  $U \setminus C$  approaching  $x$ , since then there would be an interval containing points from  $C$  with ends outside, leading to separation of the connected set  $C$  (Regularity guarantees that closure of such interval is contained in  $\bar{U}$  and hence coincides with the closure in order topology). Hence some nontrivial interval including  $x$  is contained in  $C$ .

If there simultaneously exists an increasing net approaching  $x$ , we need to repeat the reasoning to obtain that  $x$  lies in the interior of the interval contained in  $C$ .

We obtained that  $C$ , for every its point, contains an interval in a neighbourhood from the atlas of orders, which is an orderable neighbourhood in  $C$ . □

Note that without the assumption of regularity connected components may not be locally ordered. The spaces from the Examples 5.8 and 5.9 include such components. The second example is  $T_{2\frac{1}{2}}$  proving the minimality of  $T_3$  axiom.

Using the previous proposition we can prove the following property.

**Proposition 3.13.** *Every regularly locally ordered space has an open cover such that every its element is closed and either a linearly ordered space or a loop-ordered space.*

**Proof.** Denote the given regularly locally ordered space by  $X$ . Let us start with decomposing  $X$  into connected components. For every connected subset of a regularly locally ordered space is regularly locally ordered, it is either an orderable space (possibly singleton) or a loop-ordered space.

Loop-ordered component is always compact and open (Lemma 3.5).

An orderable connected component  $C$  is open unless it has an endpoint. This follows from a similar argument as presented in the proof of Lemma 3.5.

We are left with the case when  $C$  is a non-open linearly ordered component (we fix one order compatible with topology). Consider a neighbourhood

$U_x$  of a point  $x \in \partial C$  such that it is an interval in the linearly ordered set  $\overline{U_x}$  (cf. Corollary 2.9). The point  $x$  has to be an endpoint of  $C$ .

If  $C = \{x\}$  then we can separate  $(\leftarrow, x]_{\overline{U_x}}$  and  $[x, \rightarrow)_{\overline{U_x}}$  into convex and open sets and take union of those two sets which contain  $x$ . It is convex, closed in  $\overline{U_x}$  and open in  $U_x$ , hence an orderable closed and open neighbourhood of  $C$ . Observe that taking trivial separation in case when e.g.  $(\leftarrow, x]_{\overline{U_x}}$  is a singleton does still work.

If  $C \neq \{x\}$ , within  $U_x$  lies a connected set  $C_x$  such that it is a nontrivial closed interval in  $C$  containing  $x$ . The set  $C_x$  has to be convex in the order on  $U_x$ . Among the sets  $(\leftarrow, x]_{U_x}$  and  $[x, \rightarrow)_{U_x}$  precisely one contains  $C_x$ . In the other set the point  $x$  has a relative neighbourhood disjoint from  $C \setminus \{x\}$  (Lemma 3.1 implies that  $C$  is locally connected). Based on those observations we can pick an open interval  $V_x$  in  $U_x$  such that  $x \in V_x$  and among the sets  $(\leftarrow, x)_{V_x}$  and  $(x, \rightarrow)_{V_x}$  one is connected and contained in  $C$  (it is an interval of the form  $(x, y')$  for some  $y \in C_x$ ) and the other is disjoint from  $C$ . Furthermore, we can separate the interval  $V_x$  into convex and open sets and replace  $V_x$  with the part containing  $x$ . This modification does not affect the connected set  $C \cap V_x$ . Moreover, afterwards  $\partial V_x \subset C$ .

For there are only two different orders on the interval  $V_x \cap C$  compatible with topology, by reversing order on  $U_x$  if necessary, we can assume that the order on  $V_x$  agrees with the order on  $C$ .

We can take  $V := C \cup V_x \cup V_y$  where  $\{x, y\} = \partial C$ . For two endpoints  $x \neq y$  we can easily make sure that  $V_x \cap V_y = \emptyset$  (starting from disjoint  $U_x$  and  $U_y$ ). Then the equality of orders on the intersection with  $C$  is enough to define a linear order on  $V$  (a common extension) such that it is compatible with the topology. Since  $\partial V_x \subset C$  for every  $x \in \partial C$ , the set  $V$  has an empty boundary.

The proof is finished for we have shown that every connected component has a closed neighbourhood being either orderable or loop-ordered.  $\square$

### 4. Compact and similar spaces

Classification of connected regularly locally ordered spaces can be somehow extended to a wider class of spaces, namely those for which there exist *tame* atlases.

We start with noticing a following fact.

**Proposition 4.1.** *Union of an arbitrary family of loop-ordered subspaces contained in a regularly locally ordered space is both closed and open.*

**Proof.** Openness is a consequence of Lemma 3.5.

Assume  $x$  belongs to the closure of  $A := \bigcup_{i \in I} L_i$ , where  $L_i$  are loop ordered subspaces of  $X$ . Consider arbitrary ordered neighbourhood  $U$  of  $x$ . Define closed set  $F := \overline{A} \setminus U$ . Since loop ordered space cannot be embedded in a linearly ordered space,  $L_i \setminus U \neq \emptyset$  for every  $i \in I$  and consequently  $F \neq \emptyset$ . For every  $i \in I$ , the intersection  $U \cap L_i$  is a disjoint union of connected orderable sets.

By regularity we can find disjoint neighbourhoods  $V_x$  and  $V_F$  of  $x$  and  $F$ , respectively. Without loss of generality we can assume  $V_x$  is an open interval in  $U$ .

If  $x$  does not belong to  $A$ , there exists  $i_0 \in I$  such that some component of  $L_{i_0} \cap U$  is contained in  $V_x$  (such components are present in every neighbourhood of  $x$  for  $x$  is in the closure of  $A$ ). Then the endpoints of this component in  $L_{i_0}$  belong to the closure of  $V_x$  as well as to the set  $F \subseteq V_F$  what is a contradiction.  $\square$

Having the above theorem we can extend classification of regularly locally ordered spaces.

**Theorem 4.2.** *Every regularly locally ordered Lindelöf space is a disjoint union of at most countably many loop-ordered spaces and at most two orderable spaces.*

**Proof.** Denote given regularly locally ordered Lindelöf space by  $X$ . Fix a closed-open cover of  $X$  from Proposition 3.13 and choose countable subcover  $\{U_n\}_{n \in \mathbb{N}}$ . Then, by defining sets  $V_n := U_n \setminus \bigcup_{j < n} U_j$ , for  $n \in \mathbb{N}$ , we obtain a closed-open cover consisting of pairwise disjoint sets. Note that loop-ordered components were already disjoint from all other sets from the cover  $\{U_n\}_{n \in \mathbb{N}}$  hence they were not modified when passing to  $\{V_n\}_{n \in \mathbb{N}}$ . Then each of the sets  $V_n$  is either a loop-ordered space or a closed-open subspace of a linearly ordered space. Since every open subspace of a linearly ordered space is a disjoint union of orderable spaces, we have actually decomposed  $X$  into a disjoint union of at most countably many loop-ordered spaces and some number of orderable spaces.

To finish the proof it suffices to observe that arbitrary disjoint union of orderable spaces is in fact a union of at most two such spaces (Lemma 6.2 in the Appendix).  $\square$

**Corollary 4.3.** *Every second-countable regularly locally ordered space is homeomorphic to a disjoint union of at most countably many unit circles and a subspace of the real line.*

**Proof.** Such space is metrizable (see [2, 4.2.8]), hence all the loop-ordered components have to be homeomorphic to the unit circle. Second-countable linearly ordered subspaces are embeddable into the real line (see [2, 6.3.2c]).  $\square$

In the case of compact spaces the description is somehow simpler than this from Theorem 4.2.

**Corollary 4.4.** *Every compact Hausdorff locally ordered space is a disjoint union of finitely many loop-ordered spaces and possibly a single compact orderable space.*

**Proof.** Applying the theorem for Lindelöf spaces we obtain decomposition into a disjoint union of loop-ordered spaces and (at most two) linearly ordered spaces. Since each component of the union is open and the space is compact, there are finitely many of them. For each of them is closed, the linearly ordered components are compact. Disjoint union of two compact linearly ordered spaces is a compact orderable space for it is enough to treat every element of the first space as smaller than any element of the second.  $\square$

Further we can observe.

**Corollary 4.5.** *Every compact Hausdorff locally ordered space is hereditarily normal ( $T_5$ ).*

This is a special case of a more general fact on paracompact spaces.

**Lemma 4.6.** *Every paracompact Hausdorff space which admits an open cover of hereditarily normal subsets is itself hereditarily normal.*

**Proof.** Fix two separated sets  $A$  and  $B$  (i.e.  $\overline{A} \cap B = A \cap \overline{B} = \emptyset$ ) in the given paracompact space  $X$ . We will show that  $A$  has a neighbourhood with closure not intersecting  $B$ .

Given an open cover of  $X$  consisting of sets with hereditarily normal closures (for the space is  $T_3$  we can easily build such cover), we can pick a locally finite refinement  $\mathcal{V}_0$ . Now focus on  $\mathcal{V}_1 := \{V \in \mathcal{V}_0 : V \cap A \neq \emptyset\} = \{V_j\}_{j \in J}$ , an open cover of  $A$ .

For  $j \in J$ , by hereditary normality of  $\overline{V_j}$ , we can find open set  $U_j \subseteq V_j$  such that  $\overline{U_j} \cap B = \emptyset$  and  $U_j \supseteq A \cap V_j$ . Note that the collection  $\mathcal{U} := \{U_j\}_{j \in J}$  is an open cover of  $A$  locally finite in the space  $X$ .

Take  $U := \bigcup_{j \in J} U_j$ . It is a neighbourhood of  $A$ , and since  $\mathcal{U}$  is locally finite,  $\overline{U} \cap B = \bigcup_{j \in J} \overline{U_j} \cap B = \emptyset$ .  $\square$

While the lemma is interesting on its own right, we focus on the following immediate corollary.

**Corollary 4.7.** *Every paracompact Hausdorff locally ordered space is hereditarily normal ( $T_5$ ).*

*In particular every Lindelöf regularly locally ordered space is  $T_5$ .*

Since in most cases we obtain normality of a space as a consequence of compactness, paracompactness or higher separation axioms, the following question arise.

**Problem 4.8.** *Is every normal locally ordered space hereditarily normal?*

### Compact extensions and subspaces

It is well known that any linearly ordered space has a linearly ordered compactification – even one extending the original order (cf. [2, Problem 3.12.3(b)]). Regularly locally ordered spaces are precisely those locally ordered ones admitting a compactification; however, they may not admit a locally ordered compactification. In general they may not even be embeddable in a paracompact locally ordered space (see Examples 5.10 and 5.12).

Even under strong assumptions such as metrizability and local connectedness a locally ordered space may not admit a locally ordered compactification. An example can be an infinite disjoint union of unit circles. According to the Proposition 4.1, it would be closed in any bigger regularly locally ordered space hence the latter could not be compact.

In fact, from the description of compact locally ordered spaces (Corollary 4.4), one can deduce the characterisation of all spaces admitting a locally ordered compactification.

**Theorem 4.9.** *A topological space admits a locally ordered compactification if and only if it is a disjoint union of a suborderable space and finitely many loop-ordered spaces.*

**Proof.** Clearly, every subspace of a compact locally ordered space is of the above form. The reverse implication follows from the fact that a suborderable space admits a linearly ordered compactification.  $\square$

There is one more fact related to compact locally ordered spaces, namely

**Proposition 4.10.** *Every compact subset of a Hausdorff locally ordered space is a regularly locally ordered space.*

**Proof.** Let  $K$  be a compact subspace of a Hausdorff locally ordered space. Fix  $x \in K$  and denote its linearly ordered neighbourhood by  $U$ .

Consider a family  $\mathcal{V}$  of open intervals in  $U$ , such that  $\{x\} = \bigcap \mathcal{V}$  (basis of neighbourhoods for  $x$ ).  $\mathcal{V}$  is directed by inclusion. Assume the set  $\overline{V \cap K} \setminus U$  is nonempty for every  $V \in \mathcal{V}$ . The family of such compact sets has finite intersection property, hence the intersection of them all is nonempty. A point from this intersection is contained in the closure of every neighbourhood of  $x$ , what is a contradiction with the fact that the space is Hausdorff.

We obtained that for some open interval  $V \ni x$ , the compact set  $\overline{V \cap K}$  is contained in the linearly ordered subspace  $U$ , and hence it is orderable itself. Therefore,  $V \cap K$  contains an orderable neighbourhood of  $x$  in  $K$ .  $\square$

**Corollary 4.11.** *If a locally ordered Hausdorff space does not contain a loop-ordered subspace, then its every compact subset is orderable.*

## 5. Examples

The following section is a collection of all significant examples of locally ordered spaces mentioned in the paper. We also note several topological properties of presented spaces which are not main focus of this article.

When defining a locally order topology we will often use the following fact, which is a straight consequence of the axioms of topology.

**Lemma 5.1.** *For a family  $\{(X_i, \tau_i)\}_{i \in I}$  of topological spaces such that  $U \cap V \in \tau_i \cap \tau_j$ , for any two indices  $i, j \in I$  and two sets  $U \in \tau_i$  and  $V \in \tau_j$ , there exists precisely one topology  $\tau$  on  $X := \bigcup_{i \in I} X_i$  such that  $\{X_i : i \in I\}$  is an open cover of  $X$  and the induced topologies are equal to the initial.*

The presented condition means that topologies on any two spaces from the cover coincide on their intersection. In our case, where each of the spaces is linearly ordered, the condition mostly follows from the fact that the intersection is a disjoint union of open intervals with orders coinciding on each one alone.

The simple lemma presented below is a useful tool when comes to verifying semiregularity of a locally ordered space.

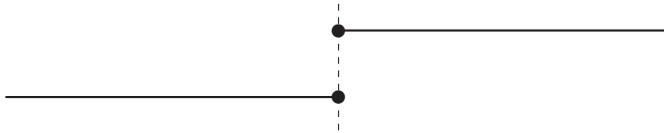
**Lemma 5.2.** *A locally ordered space is semiregular if and only if it is Hausdorff and admits an atlas of orders consisting of sets each being interior of its closure.*

To make some constructions easier to understand, we will use special graphical representation based on the following assumptions:

1. every line segment (possibly curved) denotes a set homeomorphic to an interval on the real line;
2. a neighbourhood of a point contains all close points within the horizontal line passing through;
3. a neighbourhood of a point lying on a dashed line consist of all points close to the line, except the other points lying on the line.

Symbols  $\omega$  and  $\omega_1$  stand for the first countable and the first uncountable ordinal, respectively. Then  $\omega^2$  is the ordinal isomorphic to  $\mathbb{N}^2$  with lexicographic order. We recall that  $\lambda = [0, \lambda)_\lambda$ , for any ordinal number  $\lambda$ .

**Example 5.3** (Line with doubled origin). The space of concern is obtained from the real line by adding additional point with the same deleted neighbourhoods as the point 0. The appropriate diagram is presented below.



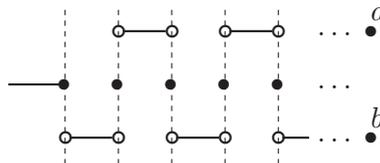
The space is  $T_1$ , second-countable, connected, locally connected, path connected but not Hausdorff nor arcwise connected. Some further properties are listed in [9].

**Example 5.4.** Consider the set  $X := (0, \infty)_\mathbb{R} \cup \{a, b\}$  ( $a \neq b, a, b \notin \mathbb{R}$ ) and the following linearly ordered spaces:

$$(0, \infty)_\mathbb{R}, \quad \bigcup_{n=1}^{\infty} (2n, 2n + 1)_\mathbb{R} \cup \{a\}, \quad \bigcup_{n=1}^{\infty} (2n - 1, 2n)_\mathbb{R} \cup \{b\},$$

where real numbers are ordered naturally and points  $a, b$  are the greatest elements in the respective sets. The topologies on the intersections coincide, hence the locally order topology is well defined.

The space  $X$  is presented in the diagram below.



1. Space  $X$  is Hausdorff but no neighbourhoods of points  $a$  and  $b$  have disjoint closures.
2. After removing points  $a$  and  $b$ , we are left with a space homeomorphic to the real line. Hence the space is  $\sigma$ -compact.
3.  $X$  is second-countable.
4. The space is semiregular, since the points from the middle row do not belong to the interiors of ordered neighbourhoods of  $a$  or  $b$ .

**Example 5.5.** Let  $a := (\omega^2, 0)$  and  $b := (\omega^2, 1)$ . Consider the sets

$$A := \omega^2 \times (1/4, 1/2)_{\mathbb{R}} \cup \{a\} \text{ and } B := \omega^2 \times (3/4, 1)_{\mathbb{R}} \cup \{b\}$$

with lexicographic orders. For each limit ordinal  $\lambda = n \cdot \omega$  ( $0 \leq n < \omega$ ) consider the set

$$U_\lambda := \lambda \times (1/2, 3/4)_{\mathbb{R}} \cup [\lambda, \lambda + \omega)_{\omega^2} \times [0, 1)_{\mathbb{R}}$$

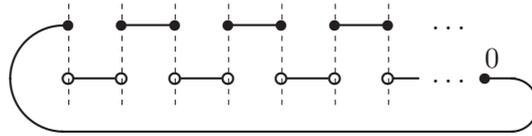
with lexicographic order. Order topologies on the intersections of given sets coincide, hence the locally order topology on  $X := \omega^2 \times [0, 1)_{\mathbb{R}} \cup \{a, b\}$  is well defined.

1. The above space  $X$  is  $T_{2\frac{1}{2}}$  and semiregular.
2.  $X$  is connected but not locally connected.
3.  $X$  is second-countable.
4.  $X$  is not completely Hausdorff.

**Proof.** Suppose  $f: X \rightarrow [0, 1]$  is continuous, and  $f(a) = 0, f(b) = 1$ . There exists  $\lambda_0 < \omega^2$  such that  $f(\{\alpha\} \times [1/4, 1/2]_{\mathbb{R}}) \subseteq [0, 1/3]_{\mathbb{R}}$  and  $f(\{\alpha\} \times [3/4, 1]_{\mathbb{R}}) \subseteq [2/3, 1]_{\mathbb{R}}$ , for any  $\alpha \in (\lambda_0, \omega^2)_{\omega^2}$ .

For a limit ordinal  $\lambda \in (\lambda_0, \omega^2)_{\omega^2}$  any neighbourhood of a point  $(\lambda, 0)$  contains a set  $\{\alpha\} \times (1/2, 3/4)_{\mathbb{R}}$  for some  $\alpha \in (\lambda_0, \lambda)_{\omega^2}$ . The values of  $f$  on the closure  $(\{\alpha\} \times [1/2, 3/4]_{\mathbb{R}})$  are then contained in arbitrarily small neighbourhood of the value  $f((\lambda, 0))$  (for sufficiently large  $\alpha$ ). This is a contradiction with  $f((\alpha, 1/2)) \leq 1/3$  and  $f((\alpha, 3/4)) \geq 2/3$ .  $\square$

**Example 5.6.** Consider the set  $X := [0, \infty)$  with topology given by the following basis of neighbourhoods: for  $x \in (0, \infty)$  we use euclidean neighbourhoods from  $(0, \infty)$  and for 0 we take sets of the form  $[0, 1/N) \cup \bigcup_{n=N}^{\infty} (2n, 2n + 1)$ , for natural  $N \geq 1$ . The diagram is following:



1. Note that after removing point 0 we are left with a space homeomorphic to the real line. Hence, the space is  $\sigma$ -compact.
2.  $X$  is second-countable and completely Hausdorff but not regular ( $T_3$ ).
3.  $X$  is arcwise connected but not locally connected.
4. The main idea behind this example is closely related to *Smirnov's deleted sequence topology* (see [9]), also referred to as *K-topology*.

**Example 5.7.** Fix an enumeration of rational numbers  $\mathbb{Q} = \{q_n\}_{n=0}^\infty$ . Consider a space being the union of the following two linearly ordered spaces:

$A := ((\mathbb{R} \setminus \mathbb{Q}) \times \{0\}) \cup \bigcup_{n=0}^\infty (\{q_n\} \times (2n, 2n + 1)_{\mathbb{R}})$  with lexicographic order inherited from  $\mathbb{R} \times \mathbb{R}$ , and  $B := \bigcup_{n=0}^\infty (\{q_n\} \times (2n, 2n + 2]_{\mathbb{R}})$  ordered by the second coordinate.

$A$  can be viewed as space  $\mathbb{R}$  modified by replacing each rational number by an open interval (homeomorphic to  $(0, 1)$ ) and  $B$  is homeomorphic to  $(0, \infty)$ . On the intersection  $A \cap B$  both topologies clearly agree, hence global topology on  $X = A \cup B$  is well defined.

1. Since the set  $B$  is dense in  $X$ , connected and separable, the whole space  $X$  is connected and separable.
2.  $X$  is not regular nor locally connected.
3.  $X$  is completely Hausdorff.
4. Consider the elements of  $(\mathbb{R} \setminus \mathbb{Q}) \times \{0\} \subseteq X$ . Neither of them belongs to the connected and dense subset  $B$ , hence removing arbitrarily many of them does not separate  $X$ .

The following two spaces contain not locally ordered connected components.

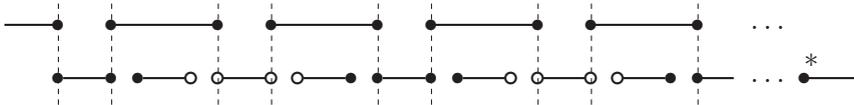
**Example 5.8.** We take the set  $X = T \cup B \cup E$ , where

$$T := \bigcup_{n=1}^{\infty} \left[ \frac{-1}{4n+1}, \frac{-1}{4n+4} \right]_{\mathbb{R}} \times \{1\}, \quad E := [0, 1]_{\mathbb{R}} \times \{0\},$$

$$B := \bigcup_{n=1}^{\infty} \left( \left[ \frac{-1}{8n}, \frac{-1}{8n+1} \right]_{\mathbb{R}} \cup \left[ \frac{-1}{8n+2}, \frac{-1}{8n+3} \right]_{\mathbb{R}} \cup \left( \frac{-1}{8n+4}, \frac{-1}{8n+5} \right)_{\mathbb{R}} \cup \left( \frac{-1}{8n+6}, \frac{-1}{8n+7} \right)_{\mathbb{R}} \right) \times \{0\}.$$

Endow the sets  $B \cup E$  and  $T \cup \bigcup_{n=2}^{\infty} \left( \frac{-1}{4n}, \frac{-1}{4n+1} \right)_{\mathbb{R}} \times \{0\}$  with order topology given by the natural order on the first coordinate. Their topologies agree on the intersection, hence the locally order topology on  $X$  is well defined.

The idea is presented on the diagram below. Note that the basic neighbourhoods of the points on the lower level ( $B \cup E$ ) do not include the points from above (i.e.  $T$  is closed).



1.  $X$  is not Hausdorff.
2. The connected component at the point  $* = (0, 0)$  does not contain the half-open intervals from  $B$ . Hence neighbourhoods of  $*$  in its component are not orderable, similarly as in the Example 2.5.

**Example 5.9.** Fix an enumeration of rationals  $\mathbb{Q} = \{q_n : n \in \mathbb{N}_+\}$  and for every irrational  $x \in \mathbb{R} \setminus \mathbb{Q}$  fix one strictly increasing sequence  $(x(k))_{k \in \mathbb{N}_+}$  of indices such that  $\lim_{k \rightarrow \infty} q_{x(k)} = x$ . Consider the following subsets of  $\mathbb{R} \times \mathbb{R}$

$$L := \bigcup_{n=1}^{\infty} \{q_n\} \times \left( \frac{-1}{2n}, \frac{-1}{2n+2} \right]_{\mathbb{R}},$$

$$P_x := \left( \{x\} \times [0, 1]_{\mathbb{R}} \right) \cup \bigcup_{k=1}^{\infty} \{q_{x(k)}\} \times \left( \frac{-1}{2x(k)}, \frac{-1}{2x(k)+1} \right)_{\mathbb{R}}, \text{ for } x \in \mathbb{R} \setminus \mathbb{Q},$$

$$D := (\mathbb{Q} \times \{0\}) \cup \bigcup_{x \in \mathbb{R} \setminus \mathbb{Q}} \{x\} \times ((0, 1]_{\mathbb{R}} \cup [2, 3]_{\mathbb{R}}).$$

Claim  $L$  and each of  $P_x$  ordered by the second coordinate and on  $D$  use lexicographic order inherited from  $\mathbb{R} \times \mathbb{R}$ . It is easy to verify that order topologies coincide on intersections of any two sheets, hence the locally order topology on  $X := L \cup D \cup \bigcup_{x \in \mathbb{R} \setminus \mathbb{Q}} P_x$  is well defined.

1.  $X$  is not  $T_3$ .
2.  $X$  is not separable.
3. It is a matter of routine to check that  $X$  is completely Hausdorff and semiregular.
4.  $X$  has a connected component which is not a locally ordered space.

**Proof.** Note that  $L$  with considered order topology is naturally homeomorphic to  $(-1, 0)$ , hence connected. Each set  $P_x$  is contained in the connected component at  $L$ . Since we can approach a rational number  $q$  with a sequence  $(x_n)$  of irrationals, we can approximate the point  $(q, 0) \in D$  with points  $(x_n, 1) \in P_{x_n} \cap D$ . Hence  $Q := \mathbb{Q} \times \{0\}$  is also contained in the same component as  $L$ . Since each of the sets  $\{x\} \times [2, 3)_{\mathbb{R}}$ , for irrational  $x$ , is both closed and open in  $X$ , they do not belong to the component at  $L$ , which then appears to be  $C := L \cup Q \cup \bigcup_{x \in \mathbb{R} \setminus \mathbb{Q}} P_x$ .

Fix a rational number  $q_0$ . We claim that the point  $(q_0, 0) \in X$  does not possess an orderable neighbourhood in  $C$ .

A small neighbourhood  $U$  of  $(q_0, 0)$  in  $C$  is contained in  $D \cap C = Q \cup \bigcup_{x \in \mathbb{R} \setminus \mathbb{Q}} \{x\} \times (0, 1]_{\mathbb{R}}$ . Moreover, every such neighbourhood consists of uncountably many copies of  $(0, 1]$  and countably many singletons (not isolated!). Actually  $q_0$  belongs to the interior of the projection  $U_1$  of  $U$  onto the first coordinate. Let  $\tilde{U}$  denote an open interval in  $\mathbb{R}$  contained in  $U_1$ .

There are only two possible orders on the open set  $\{x\} \times (0, 1]$  compatible with topology, hence a point of the form  $(x, 1)$  has to be an extremal point of its neighbourhood. Apart from at most two such points, every one has a successor or predecessor in hypothetical order inducing the topology on  $U$ . Such “neighbours” can be only points of the form  $(q, 0)$ , for rational  $q$ , or  $(x, 1)$ , for irrational  $x$ . Consider  $G$  the set of all irrational numbers  $x$  from  $\tilde{U}$ , for which there exists other irrational number  $x' \in \tilde{U}$  such that there is no element between  $(x, 1)$  and  $(x', 1)$  in the order on  $U$ . Note that  $\tilde{U} \setminus G$  is countable.

To each  $x \in G$  we can assign a positive number  $r(x) := |x - x'|$ . For any rational number  $q \in \tilde{U}$  and a sequence  $(x_n) \subseteq G$  converging to  $q$  the

values  $r(x_n)$  converge to 0, since  $(x_n, 1) \xrightarrow{U} (q, 0)$  and  $(x'_n, 1) \xrightarrow{U} (q, 0)$ . Note that every rational  $q$  is a limit of some sequence in  $G$ .

For each  $q \in \mathbb{Q} \cap \tilde{U}$  and  $N \geq 1$  let  $B(N, q)$  be a neighbourhood of  $q$  such that  $r(G \cap B(N, q)) \subseteq (0, 1/N)$ . Then the sets  $B_N := \bigcup \{B(N, q) : q \in \mathbb{Q} \cap \tilde{U}\}$ , for  $N \geq 1$ , have the intersection contained in  $\tilde{U} \setminus G$ . This intersection would be a countable dense  $G_\delta$  subset of  $\tilde{U}$ . From the Baire theorem we obtain contradiction.  $\square$

The last three examples are well known (though not necessarily for being locally ordered) but we describe them briefly here to make this bank of examples more complete. For more details on them the reader is referred to [9].

**Example 5.10** (Rational sequence topology). For every  $x \in \mathbb{R} \setminus \mathbb{Q}$  fix one sequence of rationals  $(x_i)_{i \in \mathbb{N}}$  convergent to  $x$ . Consider set  $X := \mathbb{R}$  with the following topology: for  $x \in \mathbb{Q}$  the singleton  $\{x\}$  is open, while for  $x \in \mathbb{R} \setminus \mathbb{Q}$  the basic neighbourhoods have the form  $\{x_i : i \geq n\} \cup \{x\}$ , where  $n \in \mathbb{N}$ .

1. All basic neighbourhoods are clearly closed and orderable hence the space is regularly locally ordered. It is also locally compact.
2. The presented topology is finer than the standard topology on  $\mathbb{R}$ . Hence, the diagonal  $(\Delta_X)$  is a  $G_\delta$  subset of the product space.
3. The space  $X$  is not normal ( $T_4$ ). It can be proven by Jones' Lemma (see e.g. [11]) since the space is separable and contains a closed discrete subset  $(\mathbb{R} \setminus \mathbb{Q})$  of cardinality continuum. According to Corollary 4.5 it does not admit a locally ordered compactification.
4. A curious observation is that on each of the ordered neighbourhoods we can consider natural order induced from  $\mathbb{R}$ , obtaining that for any two sheets from the atlas the orders coincide on the intersection. It shows that the existence of such "neat" atlas does not imply any further "regularity" of a locally ordered space. See also the next example.

**Example 5.11** (Pointed rational extension of  $\mathbb{R}$ ). For each  $x \in \mathbb{R} \setminus \mathbb{Q}$  take the set  $\{x\} \cup \mathbb{Q}$  with the order topology inherited from  $\mathbb{R}$ . Such covering defines a locally order topology on the set  $X := \mathbb{R}$ , since intersection of any two sheets equals  $\mathbb{Q}$  (with standard topology) and is open in both.

1. The presented topology is finer than the standard topology on  $\mathbb{R}$ . Hence  $X$  is completely Hausdorff.
2.  $X$  is separable and first-countable but not second-countable.
3. One can notice that in the described space the only connected sets are intervals with respect to the classical order on  $\mathbb{R}$  and the whole space is connected. However, no point has a connected orderable neighbourhood.
4.  $X$  is not semiregular. Picking any open interval in  $\mathbb{Q}$ , its closure would be an interval in  $\mathbb{R}$  and taking interior does not exclude all the irrationals.
5. By adding additional point “ $\infty$ ” one can close the space into a “circle”. Then there would be no cut-point.

**Example 5.12** (Dieudonné plank). Let  $X := [0, \omega] \times [0, \omega_1] \setminus \{(\omega, \omega_1)\}$ . For every  $n \in \omega$  and  $\alpha \in \omega_1$  we declare the following sets open:

$$\{(n, \alpha)\}, \quad U_{n,\alpha} := \{(m, \alpha) : m \in [n, \omega]\}, \quad V_{n,\alpha} := \{(n, \beta) : \beta \in [\alpha, \omega_1]\}.$$

The family of all such sets is closed on intersections, hence it is a basis of a topology.

1. Considered topology on  $X$  is locally ordered, since every set  $U_{n,\alpha}$  is naturally homeomorphic to  $[n, \omega]$ , and every set  $V_{n,\alpha}$  is homeomorphic to the set  $\omega_1 \times \mathbb{Z} \cup \{(\omega_1, 0)\}$  with the topology given by the lexicographic order. Every set from the basis is closed, and therefore,  $X$  is regularly locally ordered.
2.  $X$  is not locally compact.
3.  $X$  is not normal, since the closed sets  $A := \omega \times \{\omega_1\}$  and  $B := \{\omega\} \times \omega_1$  do not admit disjoint neighbourhoods. It can be observed that any neighbourhood of  $A$  has at most countable complement in  $\omega \times \omega_1$ , while every neighbourhood of  $B$  has uncountable intersection with  $\omega \times \omega_1$ .  
Furthermore,  $X$  does not admit a locally ordered compactification.
4.  $X$  is not separable nor first-countable.

**Remark 5.13.** In [3] Herrlich mentioned the space known as “Deleted Tychonoff plank” (see [9]) as an example of not normal regularly locally ordered space. We replaced the example with Dieudonné plank, since its local orderability is more explicit.

## 6. Appendix

The below theorem is very well known, but the presented topological proof seems to be worth mentioning.

**Theorem 6.1.** *For every connected linearly ordered space consisting of at least two points there exist exactly two linear orders compatible with topology.*

**Proof.** Let  $(X, L)$  be a linearly ordered set ( $L$  stands for the order as a relation, i.e.  $L = \{(x, y) : x <_{(X,L)} y\}$ ) such that the induced order topology is connected. The space  $X \times X$  with the product topology is then also connected. Moreover, from connexity,  $X \times X \setminus \Delta_X = L \cup L^{-1}$ , where  $L^{-1} = \{(x, y) : x >_{(X,L)} y\}$ .

Let  $R \subseteq X \times X$  be a linear order compatible with topology. For two arbitrary points  $(x_1, y_1)$  and  $(x_2, y_2)$  in  $R$ , the sets  $(\leftarrow, x_1]_R \times [y_1, \rightarrow)_R$  and  $(\leftarrow, x_2]_R \times [y_2, \rightarrow)_R$  are contained in  $R$  and connected. Their intersection is not empty, since it contains the point  $(\min_R(x_1, x_2), \max_R(y_1, y_2))$ . Hence  $R$  is a connected subset of the product space and has to be contained in one of the disjoint open sets  $L$  and  $L^{-1}$ . Since no proper subset of a linear order is a linear order, either  $R = L$  or  $R = L^{-1}$ . □

**Lemma 6.2.** *If a topological space is a disjoint union of an arbitrary family of totally ordered spaces, then it can be represented as a disjoint union of at most two linearly ordered spaces.*

**Proof.** By concatenation of two given linearly ordered spaces we mean extending both orders to an order on the union in such a way that all elements of the first space are smaller than elements of the second. To preserve the disjointness of the union we must be sure that the first space contains the greatest element if and only if the second one has the smallest.

Let  $\mathcal{U}$  be a given family of linearly ordered spaces. We divide it into three disjoint subfamilies  $\mathcal{U}_0$ ,  $\mathcal{U}_1$  and  $\mathcal{U}_2$ , namely spaces with no extremal point, with one extremal point (smallest or greatest element) and with both extremal points respectively.

We can easily concatenate any two spaces from  $\mathcal{U}_1$  to obtain one linearly ordered space with no endpoint (reversing one of the orders, if necessary). Proceeding this way we can make sure that there is at most one element in  $\mathcal{U}_1$ .

Similarly, we can concatenate countably many spaces from  $\mathcal{U}_2$  into one space with no endpoint, hence we may reduce to the case when the family

$\mathcal{U}_2$  consists of at most one element (any finite concatenation still has both extremal points).

The family  $\mathcal{U}_0$  (under no assumptions on cardinality) can be also concatenated to obtain one linearly ordered space by using sufficiently large ordinal. We are left in the case when all three families are at most singletons. Since space with one extremal point can be easily concatenated (after reversing the order, if necessary) with any linearly ordered space, we are done.  $\square$

Herrlich formulated a similar, but more specific lemma providing sufficient conditions for orderability of a disjoint union [3, IV. Hilfssatz 4.].

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