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SOME CARDINAL CHARACTERISTICS RELATED TO THE COVERING NUMBER AND THE UNIFORMITY OF THE MEAGRE IDEAL

A b s t r a c t. We extend the concepts of splitting, reaping, and independent families to families of functions and permutations on ω and define associated cardinal characteristics $\mathfrak{s}_f$, $\mathfrak{s}_p$, $\mathfrak{r}_f$, $\mathfrak{r}_p$, $\mathfrak{i}_f$, and $\mathfrak{i}_p$. We study relationships among $\text{cov}(\mathcal{M})$, $\text{non}(\mathcal{M})$, and these cardinals. In this paper, we show that $\mathfrak{s}_f = \text{non}(\mathcal{M}) = \mathfrak{s}_p$, $\mathfrak{r}_f = \text{cov}(\mathcal{M}) \leq \mathfrak{r}_p$, and $\text{cov}(\mathcal{M}) \leq \mathfrak{i}_f, \mathfrak{i}_p$.

1. Introduction

The *covering number* of the meagre ideal $\mathcal{M}$, $\text{cov}(\mathcal{M})$, is the smallest size of a family of meagre subsets of ${}^\omega\omega$ whose union is ${}^\omega\omega$ and the *uniformity*

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of $\mathcal{M}$, $\text{non}(\mathcal{M})$, is the smallest size of a non-meagre subset of ${}^\omega\omega$ (see [3] or [7, Chapter III] for more details). It is well-known that $\aleph_1 \leq \mathfrak{p} \leq \text{cov}(\mathcal{M}) \leq \mathfrak{r} \leq \mathfrak{i} \leq \mathfrak{c}$ and $\aleph_1 \leq \mathfrak{p} \leq \mathfrak{s} \leq \text{non}(\mathcal{M}) \leq \mathfrak{c}$, where $\mathfrak{p}$, $\mathfrak{s}$, $\mathfrak{r}$, and $\mathfrak{i}$ are the *pseudo-intersection*, the *splitting*, the *reaping*, and the *independence numbers* respectively (for more details about these numbers see [3] or [6, Chapter 9]).

The *almost disjoint number* $\mathfrak{a}$ is the smallest size of a maximal almost disjoint family of infinite subsets of ω . It has been shown that both $\mathfrak{a}$ and $\text{non}(\mathcal{M})$ lie between the *bounding number* $\mathfrak{b}$ and $\mathfrak{c}$ (see [3] and [6]). Almost disjoint families of functions and permutations on ω and associated cardinal characteristics, denoted by $\mathfrak{a}_e$ and $\mathfrak{a}_p$ respectively, were studied by Zhang in [9]. Brendle, Spinas, and Zhang showed in [4] that $\text{non}(\mathcal{M})$ is a lower bound of both $\mathfrak{a}_e$ and $\mathfrak{a}_p$ (cf. [4, Theorem 2.2 and Proposition 4.6]).

Independent families of functions and permutations on ω and associated cardinal characteristics $\mathfrak{i}_f$ and $\mathfrak{i}_p$ were studied by us in [8]. We have shown that $\mathfrak{p} \leq \mathfrak{i}_f, \mathfrak{i}_p \leq \mathfrak{i}$ and also mentioned that $\text{cov}(\mathcal{M})$ is a lower bound of both $\mathfrak{i}_f$ and $\mathfrak{i}_p$. In this paper, we give a full direct proof of this fact.

We also extend the concepts of splitting and reaping families to families of functions and permutations on ω and define associated cardinal characteristics $\mathfrak{s}_f$, $\mathfrak{s}_p$, $\mathfrak{r}_f$, and $\mathfrak{r}_p$. We study relationships among $\text{cov}(\mathcal{M})$, $\text{non}(\mathcal{M})$, and these cardinals. As mentioned above, $\mathfrak{s} \leq \text{non}(\mathcal{M})$ and $\text{cov}(\mathcal{M}) \leq \mathfrak{r}$. In this paper, we show that $\mathfrak{s}_f = \text{non}(\mathcal{M}) = \mathfrak{s}_p$ and $\mathfrak{r}_f = \text{cov}(\mathcal{M}) \leq \mathfrak{r}_p$.

2. Splitting and reaping families

A set $A \subseteq \omega$ *splits* an infinite set $B \subseteq \omega$ if both $B \cap A$ and $B \setminus A$ are infinite. A *splitting family* $\mathcal{S}$ is a family of infinite subsets of ω such that each infinite set $B \subseteq \omega$ is split by at least one $A \in \mathcal{S}$. A *reaping family* $\mathcal{R}$ is a family of infinite subsets of ω such that there is no infinite subset of ω which splits every member of $\mathcal{R}$. The *splitting number* $\mathfrak{s}$ is the smallest cardinality of any splitting family and the *reaping number* $\mathfrak{r}$ is the smallest cardinality of any reaping family.

We write ${}^\omega\omega$ and $\text{Sym}(\omega)$ for the set of functions and the set of permutations, respectively, on ω . We extend the concepts of splitting and reaping families to families of functions and permutations on ω . To be precise, we say $f \in {}^\omega\omega$ splits $g \in {}^\omega\omega$ if both $g \cap f$ and $g \setminus f$ are infinite. A *splitting family*

$\mathcal{S}$ of functions (permutations) is a family of functions (permutations) on ω such that each $g \in {}^\omega\omega$ ($g \in \text{Sym}(\omega)$) is split by an $f \in \mathcal{S}$. A *reaping family* $\mathcal{R}$ of functions (permutations) is a family of functions (permutations) on ω such that there is no function (permutation) on ω which splits every member of $\mathcal{R}$. We define corresponding cardinal characteristics $\mathfrak{s}_f$, $\mathfrak{s}_p$, $\mathfrak{r}_f$, and $\mathfrak{r}_p$ as follows.

$$\begin{aligned}\mathfrak{s}_f &= \min\{|\mathcal{S}| : \mathcal{S} \subseteq {}^\omega\omega \text{ is a splitting family}\}, \\ \mathfrak{s}_p &= \min\{|\mathcal{S}| : \mathcal{S} \subseteq \text{Sym}(\omega) \text{ is a splitting family}\}, \\ \mathfrak{r}_f &= \min\{|\mathcal{R}| : \mathcal{R} \subseteq {}^\omega\omega \text{ is a reaping family}\}, \text{ and} \\ \mathfrak{r}_p &= \min\{|\mathcal{R}| : \mathcal{R} \subseteq \text{Sym}(\omega) \text{ is a reaping family}\}.\end{aligned}$$

It is easy to see that the above definitions are well-defined since ${}^\omega\omega$ and $\text{Sym}(\omega)$ are splitting and reaping families of functions and permutations respectively.

First, we shall show that $\mathfrak{s}_f = \text{non}(\mathcal{M})$ and $\mathfrak{r}_f = \text{cov}(\mathcal{M})$. The following is Theorem 5.9 in [3]. The first statement is also from [1, Corollary 1.8].

Theorem 2.1.

$$\begin{aligned}\text{cov}(\mathcal{M}) &= \min\{|\mathcal{C}| : \mathcal{C} \subseteq {}^\omega\omega \wedge \neg\exists f \in {}^\omega\omega \forall g \in \mathcal{C} (f \cap g \text{ is infinite})\}, \text{ and} \\ \text{non}(\mathcal{M}) &= \min\{|\mathcal{C}| : \mathcal{C} \subseteq {}^\omega\omega \wedge \forall f \in {}^\omega\omega \exists g \in \mathcal{C} (f \cap g \text{ is infinite})\}.\end{aligned}$$

Theorem 2.2. $\mathfrak{s}_f = \text{non}(\mathcal{M})$ and $\mathfrak{r}_f = \text{cov}(\mathcal{M})$.

Proof. It follows immediately from the above theorem that $\mathfrak{r}_f \leq \text{cov}(\mathcal{M})$ and $\text{non}(\mathcal{M}) \leq \mathfrak{s}_f$. To show that $\mathfrak{s}_f \leq \text{non}(\mathcal{M})$, let $\mathcal{C} \subseteq {}^\omega\omega$ be an infinite family such that for all $f \in {}^\omega\omega$, there exists a $g \in \mathcal{C}$ such that $f \cap g$ is infinite.

For each $g \in \mathcal{C}$, define $\tilde{g} \in {}^\omega\omega$ by

$$\tilde{g}(n) = \begin{cases} g(n) & \text{if } n \text{ is even,} \\ g(n) + 1 & \text{if } n \text{ is odd.} \end{cases}$$

Let $\mathcal{D} = \mathcal{C} \cup \{\tilde{g} : g \in \mathcal{C}\}$. To show that $\mathcal{D}$ is a splitting family, let $f \in {}^\omega\omega$. By the property of $\mathcal{C}$, there is a $g \in \mathcal{C}$ such that $f \cap g$ is infinite. If $f \setminus g$ is finite, then there is an $n_0 < \omega$ such that $f(n) = g(n)$ for all $n \geq n_0$, and hence $\tilde{g}$ splits f . Otherwise, g splits f . Thus $\mathfrak{s}_f \leq |\mathcal{D}| = |\mathcal{C}|$. Since $\mathcal{C}$ is arbitrary, $\mathfrak{s}_f \leq \text{non}(\mathcal{M})$.

To show that $\text{cov}(\mathcal{M}) \leq \mathfrak{r}_f$, let $\mathcal{C} \subseteq {}^\omega\omega$ be an infinite family such that $|\mathcal{C}| < \text{cov}(\mathcal{M})$. We shall show that $\mathcal{C}$ is not a reaping family.

For each $g \in \mathcal{C}$, let $g \oplus 1 \in {}^\omega\omega$ be defined by $(g \oplus 1)(n) = g(n) + 1$. Let $\mathcal{D} = \mathcal{C} \cup \{g \oplus 1 : g \in \mathcal{C}\}$. Then $|\mathcal{D}| = |\mathcal{C}| < \text{cov}(\mathcal{M})$. By the above theorem, there is an $f \in {}^\omega\omega$ such that $f \cap h$ is infinite for any $h \in \mathcal{D}$. Consider a $g \in \mathcal{C}$. Since $f \cap (g \oplus 1)$ is infinite, there are infinitely many $k \in \omega$ such that $f(k) \neq g(k)$. Hence $g \setminus f$ is infinite. Since $f \cap g$ is infinite, f splits g . Therefore, $\mathcal{C}$ is not a reaping family. $\square$

Next, we shall show that $\text{cov}(\mathcal{M}) \leq \mathfrak{r}_p$. The proofs make use of Martin's Axiom. We start with some relevant definitions and known facts.

Definition 2.3. $MA_{\mathbb{P}}(\kappa)$ is the statement that whenever $\mathcal{D}$ is a family of dense subsets of a poset $\mathbb{P}$ with $|\mathcal{D}| \leq \kappa$, there exists a filter G on $\mathbb{P}$ such that $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.

By the Generic Filter Existence Lemma [7, Lemma III.3.14], we obtain the following theorem.

Theorem 2.4. $MA_{\mathbb{P}}(\kappa)$ holds for any poset $\mathbb{P}$ and $\kappa \leq \aleph_0$.

Definition 2.5. A subset C of a poset $\mathbb{P}$ is *centered* if, for any $n \in \omega$ and any $p_1, p_2, \dots, p_n \in C$ there is a $q \in \mathbb{P}$ such that $q \leq p_i$ for all i . $\mathbb{P}$ is *σ -centered* if $\mathbb{P}$ is a countable union of centered subsets of $\mathbb{P}$.

Definition 2.6. $\mathfrak{m}_\sigma$ is the least κ such that there is a σ -centered poset $\mathbb{P}$ for which $MA_{\mathbb{P}}(\kappa)$ fails, and $\mathfrak{m}_{\text{ctbl}}$ is the least κ such that there is a countable poset $\mathbb{P}$ for which $MA_{\mathbb{P}}(\kappa)$ fails.

We have shown, in Theorem 2.2, that $\mathfrak{r}_f = \text{cov}(\mathcal{M})$. Now, we show that $\text{cov}(\mathcal{M}) \leq \mathfrak{r}_p$ by using the following theorem which is Proposition (d) in [5].

Theorem 2.7. $\mathfrak{m}_{\text{ctbl}} = \text{cov}(\mathcal{M})$.

Theorem 2.8. $\text{cov}(\mathcal{M}) \leq \mathfrak{r}_p$.

Proof. It suffices to show that $\mathfrak{m}_{\text{ctbl}} \leq \mathfrak{r}_p$. To show this, let $\mathcal{C} \subseteq \text{Sym}(\omega)$ be such that $\aleph_0 \leq |\mathcal{C}| < \mathfrak{m}_{\text{ctbl}}$. Consider the poset $\mathbb{P} = \text{Fn}_{1-1}(\omega, \omega)$, i.e. $\{s \subseteq \omega \times \omega : s \text{ is a finite injection}\}$. For each $n \in \omega$ and $f \in \mathcal{C}$, let

$$\begin{aligned} A_n &= \{p \in \mathbb{P} : n \in \text{dom}(p) \cap \text{ran}(p)\}, \\ B_{n,f} &= \{p \in \mathbb{P} : \exists k \geq n \exists \ell \geq n (p(k) = f(k) \wedge p(\ell) \neq f(\ell))\}. \end{aligned}$$

Then A_n and $B_{n,f}$ are dense in $\mathbb{P}$ for any $n \in \omega$ and $f \in \mathcal{C}$. Let

$$\mathcal{D} = \{A_n : n \in \omega\} \cup \{B_{n,f} : n \in \omega, f \in \mathcal{C}\}.$$

Since $\mathcal{D}$ is of size $< \mathfrak{m}_{\text{ctbl}}$, there is a filter G on $\mathbb{P}$ such that $G \cap A_n \neq \emptyset \neq G \cap B_{n,f}$ for any $n \in \omega$ and $f \in \mathcal{C}$. Let $g = \bigcup G$. Then $g \in \text{Sym}(\omega)$ and for any $n \in \omega$ and any $f \in \mathcal{C}$, we have that $g(k) = f(k)$ and $g(\ell) \neq f(\ell)$ for some $k, \ell \geq n$. Hence for any $f \in \mathcal{C}$, $f \cap g$ and $f \setminus g$ are infinite, so g splits f . Thus $\mathcal{C}$ is not a reaping family. $\square$

It is well-known that $\mathfrak{p} \leq \mathfrak{s}$ (cf. [6, Chapter 9]). Now, we shall use the fact below to show that $\mathfrak{p}$ is also a lower bound of $\mathfrak{s}_p$. The following theorem is from Bell ([2]), and is also Theorem III.3.61 in [7].

Theorem 2.9. $\mathfrak{m}_\sigma = \mathfrak{p}$.

Theorem 2.10. $\mathfrak{p} \leq \mathfrak{s}_p$.

Proof. It suffices to show that $\mathfrak{m}_\sigma \leq \mathfrak{s}_p$. To show this, let $\mathcal{C} \subseteq \text{Sym}(\omega)$ be such that $\aleph_0 \leq |\mathcal{C}| < \mathfrak{m}_\sigma$. Define the poset $\mathbb{P} = \text{Fn}_{1-1}(\omega, \omega) \times [\mathcal{C}]^{<\omega}$, where $(s, E) \leq (t, F)$ if and only if

$$s \supseteq t, E \supseteq F \text{ and } \forall n \in \text{dom}(s) \setminus \text{dom}(t) \forall f \in F (s(n) \neq f(n)).$$

Clearly this poset is σ -centered, as the set $\{(s, E) \in \mathbb{P} : E \in [\mathcal{C}]^{<\omega}\}$ is centered for any fixed s and $\text{Fn}_{1-1}(\omega, \omega)$ is countable. For each $n \in \omega$ and $f \in \mathcal{C}$, let

$$\begin{aligned} A_n &= \{(s, E) \in \mathbb{P} : n \in \text{dom}(s) \cap \text{ran}(s)\}, \\ B_f &= \{(s, E) \in \mathbb{P} : f \in E\}. \end{aligned}$$

It is easy to see that B_f is dense in $\mathbb{P}$ for all $f \in \mathcal{C}$. To show that A_n is dense in $\mathbb{P}$ for any $n \in \omega$, let $n \in \omega$ and $(s, E) \in \mathbb{P}$. Since s is a finite function and E is a finite set of injections, we can pick $k \in \omega \setminus \text{dom}(s)$ and $\ell \in \omega \setminus \text{ran}(s)$ so that $(k, n), (n, \ell) \notin \bigcup E$. We choose

$$t = \begin{cases} s & \text{if } n \in \text{dom}(s) \cap \text{ran}(s), \\ s \cup \{(k, n)\} & \text{if } n \in \text{dom}(s) \setminus \text{ran}(s), \\ s \cup \{(n, \ell)\} & \text{if } n \in \text{ran}(s) \setminus \text{dom}(s), \\ s \cup \{(k, n), (n, \ell)\} & \text{if } n \notin \text{dom}(s) \cup \text{ran}(s). \end{cases}$$

Then $(t, E) \leq (s, E)$ where $(t, E) \in A_n$. So A_n is dense in $\mathbb{P}$. Let

$$\mathcal{D} = \{A_n : n \in \omega\} \cup \{B_f : f \in \mathcal{C}\}.$$

Since $\mathcal{D}$ is of size $|\mathcal{C}| < \mathfrak{m}_\sigma$, there is a filter G on $\mathbb{P}$ such that $G \cap A_n \neq \emptyset \neq G \cap B_f$ for any $n \in \omega$ and $f \in \mathcal{C}$. Let $g = \bigcup \text{dom}(G)$. Then $g \in \text{Sym}(\omega)$.

To show that $g \cap f$ is finite for any $f \in \mathcal{C}$, let $f \in \mathcal{C}$. Since $G \cap B_f \neq \emptyset$, there is a $(s, E) \in G$ such that $f \in E$. Let $m \in \text{dom}(g) \setminus \text{dom}(s)$. We shall show that $g(m) \neq f(m)$. Since $(m, g(m)) \in g = \bigcup \text{dom}(G)$, there is a $(t, F) \in G$ such that $(m, g(m)) \in t$. Since G is a filter, there is a $(s', E') \in G$ such that $(s', E') \leq (s, E)$ and $(s', E') \leq (t, F)$. Then $m \in \text{dom}(s') \setminus \text{dom}(s)$ and hence $g(m) = t(m) = s'(m) \neq f(m)$. Therefore, $g(m) \neq f(m)$ for any $m \in \text{dom}(g) \setminus \text{dom}(s)$. So $\{m : g(m) = f(m)\} \subseteq \text{dom}(s)$, which implies that $g \cap f$ is finite. Therefore, $\mathcal{C}$ is not a splitting family. $\square$

The above proof shows the relationship between $\mathfrak{p}$ and $\mathfrak{s}_p$ by using the fact that $\mathfrak{m}_\sigma = \mathfrak{p}$. However, since $\mathfrak{p} \leq \mathfrak{s} \leq \text{non}(\mathcal{M})$, a stronger result can be obtained as shown in the following theorem. The notation $\exists^\infty n$ means “there are infinitely many” and $\forall^\infty n$ means “for all but finitely many”.

Theorem 2.11. $\text{non}(\mathcal{M}) = \mathfrak{s}_p$.

Proof. We first show that $\mathfrak{s}_p \leq \text{non}(\mathcal{M})$. Note that $\text{Sym}(\omega)$ is homeomorphic to ${}^\omega\omega$, so the notion of “the smallest size of a meagre set” in both (topological) spaces are the same. Let $\mathcal{S} \subseteq \text{Sym}(\omega)$ be such that $|\mathcal{S}| < \mathfrak{s}_p$. We shall show that $\mathcal{S}$ is meagre in $\text{Sym}(\omega)$. By the definition of $\mathfrak{s}_p$, there is a $g \in \text{Sym}(\omega)$ such that, for each $f \in \mathcal{S}$, $\forall^\infty n [g(n) \neq f(n)]$ or $\forall^\infty n [g(n) = f(n)]$. Let $\mathcal{S}_0 = \{f \in \mathcal{S} : \forall^\infty n [g(n) \neq f(n)]\}$. We claim that $\mathcal{S}_0$ is meagre in $\text{Sym}(\omega)$. For each $n < \omega$, let

$$C_n = \{f \in \text{Sym}(\omega) : \forall m > n [g(m) \neq f(m)]\}.$$

It is straightforward to show that C_n is closed nowhere dense and $\mathcal{S}_0 \subseteq \bigcup_{n < \omega} C_n$, and hence $\mathcal{S}_0$ is meagre. Since $\mathcal{S} \setminus \mathcal{S}_0 = \{f \in \mathcal{S} : \forall^\infty n [g(n) = f(n)]\}$ is countable (and hence is meagre), $\mathcal{S} = \mathcal{S}_0 \cup (\mathcal{S} \setminus \mathcal{S}_0)$ is meagre.

We next show that $\text{non}(\mathcal{M}) \leq \mathfrak{s}_p$. Let $\mathcal{S} \subseteq \text{Sym}(\omega)$ be such that $|\mathcal{S}| < \text{non}(\mathcal{M})$, and we shall show that $\mathcal{S}$ is not a splitting family.

Claim. There exists an injection $f \in {}^\omega\omega$ such that $f(n) > n$ for all $n < \omega$ and for all $q \in \mathcal{S}$, $\forall^\infty n [f(n) \neq q(n)]$ and $\forall^\infty n [f(n) \neq q^{-1}(n)]$.

Proof. Let $\pi : \omega^2 \rightarrow \omega$ be a one-to-one map such that $\pi(n, m) > n$ for all $n, m < \omega$. For any $q \in \text{Sym}(\omega)$, we define $q^+ \in {}^\omega\omega$ by

$$q^+(n) = \begin{cases} m & \text{if } q(n) = \pi(n, m), \\ 0 & \text{otherwise.} \end{cases}$$

Put $\mathcal{S}^{-1} = \{p^{-1} : p \in \mathcal{S}\}$ and $\mathcal{S}^+ = \mathcal{S} \cup \mathcal{S}^{-1} \cup \{q^+ : q \in \mathcal{S} \cup \mathcal{S}^{-1}\}$. Since $|\mathcal{S}| < \text{non}(\mathcal{M})$, by Theorem 2.1, there exists an $\hat{f} \in {}^\omega\omega$ such that for all $g \in \mathcal{S}^+$, $\forall^\infty n [\hat{f}(n) \neq g(n)]$. In particular, for each $q \in \mathcal{S} \cup \mathcal{S}^{-1}$, $\forall^\infty n [\hat{f}(n) \neq q^+(n)]$. Define $f \in {}^\omega\omega$ by $f(n) = \pi(n, \hat{f}(n))$. Clearly f is one-to-one and $f(n) > n$ for all $n < \omega$. Notice that

$$\forall q \in \mathcal{S} \cup \mathcal{S}^{-1} \forall n < \omega [f(n) = q(n) \rightarrow \hat{f}(n) = q^+(n)].$$

Hence, for each $q \in \mathcal{S} \cup \mathcal{S}^{-1}$, $\forall^\infty n [f(n) \neq q(n)]$, and the proof of the claim is complete.

Let $f(k) = n_k$, and note that $n_k > k$ for all k and n_k 's are distinct. Define $p \in \text{Sym}(\omega)$ recursively as follows. Suppose we have already defined $p \upharpoonright k$. If there exists an $i < k$ such that $p(i) = k$ then put $p(k) = i$; otherwise, put $p(k) = n_k$. Note that, after the construction is done, if $p(x) = y$ then $(x, y) = (k, n_k)$ or $(x, y) = (n_k, k)$ for some k . So $p(p(x)) = x$ for all $x < \omega$, and hence p is bijective.

We finally show that $\forall^\infty k [p(k) \neq q(k)]$ for all $q \in \mathcal{S}$. Suppose to the contrary that there is a $q \in \mathcal{S}$ such that $\exists^\infty k [p(k) = q(k)]$. Let $X = \{k : p(k) = n_k\}$. Note that $\omega \setminus X = \{n_k : p(n_k) = k\}$. Then either

$$\exists^\infty k \in X [p(k) = q(k)] \text{ or } \exists^\infty i \in \omega \setminus X [p(i) = q(i)].$$

In the former case, we have $\exists^\infty k [f(k) = n_k = p(k) = q(k)]$. In the latter case, we have $\exists^\infty k [k = p(n_k) = q(n_k)]$, so $\exists^\infty k [q^{-1}(k) = n_k = f(k)]$. Both cases contradict the above claim. Therefore $\mathcal{S}$ is not a splitting family. $\square$

3. Independent families

An infinite set $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is said to be an *independent family* (or shortly i.f.) if, for any disjoint finite sets $A, B \subseteq \mathcal{I}$, $\bigcap A \setminus \bigcup B$ is infinite. We interpret $\bigcap \emptyset = \omega$. The cardinal $\mathfrak{i}$ is defined as the least cardinality of a maximal independent family. We extend the concept of independent families to families of functions and permutations on ω and define corresponding cardinal characteristics $\mathfrak{i}_f$ and $\mathfrak{i}_p$ as follows.

$$\begin{aligned} \mathfrak{i}_f &= \min\{|\mathcal{I}| : \mathcal{I} \subseteq {}^\omega\omega \text{ is a maximal independent family}\} \text{ and} \\ \mathfrak{i}_p &= \min\{|\mathcal{I}| : \mathcal{I} \subseteq \text{Sym}(\omega) \text{ is a maximal independent family}\}. \end{aligned}$$

Since there is an i.f. of permutations of cardinality $\mathfrak{c}$ (see Proposition 2.1 in [8]), $\mathfrak{i}_f$ and $\mathfrak{i}_p$ are well-defined.

We shall show that $\text{cov}(\mathcal{M})$ is a lower bound of $\mathfrak{i}_p$ (and also $\mathfrak{i}_f$). First, we need the following fact.

Fact. $\text{cov}(\mathcal{M})$ is the least cardinality of a family of open dense subsets of ${}^\omega\omega$ whose intersection is empty.

This fact follows from Proposition (a) in [5] by viewing a Polish space X as the Baire space ${}^\omega\omega$ (with the basic open sets of the form $[p] = \{f \in {}^\omega\omega : f \supseteq p\}$ for $p \in {}^{<\omega}\omega$) together with the fact from topology that “a subset O of a topological space X is open dense if and only if $X \setminus O$ is closed nowhere-dense”. Note the fact that $D \subseteq \mathbb{P}$ is dense in the Cohen poset $\mathbb{P} = {}^{<\omega}\omega$ if and only if $[D] = \{f \in {}^\omega\omega : f \supseteq p \text{ for some } p \in D\}$ is open dense in the Baire space ${}^\omega\omega$.

For an infinite family $\mathcal{C} \subseteq {}^\omega\omega$, let

$$\text{bc}(\mathcal{C}) = \left\{ \bigcap A \setminus \bigcup B : A, B \in \text{fin}(\mathcal{C}), A \cap B = \emptyset \text{ and } A \neq \emptyset \right\}.$$

Then each member of $\text{bc}(\mathcal{C})$ is a function and is an injection if $\mathcal{C}$ is a family of permutations. Notice that $\mathcal{C}$ is an independent family if and only if every member of $\text{bc}(\mathcal{C})$ is infinite.

Theorem 3.1. $\text{cov}(\mathcal{M}) \leq \mathfrak{i}_p$.

Proof. Let $\mathcal{C} \subseteq \text{Sym}(\omega)$ be an i.f. of permutations such that $\aleph_0 \leq |\mathcal{C}| < \text{cov}(\mathcal{M})$. We shall show that $\mathcal{C}$ is not maximal.

For any $p \in {}^{<\omega}\omega \cup {}^\omega\omega$, let $\hat{p}$ be the one-to-one sequence obtained from p by removing all repetitions of each occurrence of $p(i)$ except its first one. Let $\mathbb{P} = {}^{<\omega}\omega$ and for each $x \in \text{bc}(\mathcal{C})$ and $n < \omega$, let

$$\begin{aligned} D_{x,n} &= \{p \in \mathbb{P} : \exists k, \ell \geq n (k, \ell \in \text{dom}(\hat{p}) \cap \text{dom}(x) \wedge \hat{p}(k) = x(k) \\ &\quad \wedge \hat{p}(\ell) \neq x(\ell))\}, \\ A_n &= \{p \in \mathbb{P} : n \in \text{ran}(p)\}. \end{aligned}$$

It is easy to see that each A_n is dense in $\mathbb{P}$. To show that each $D_{x,n}$ is dense in $\mathbb{P}$, let $x \in \text{bc}(\mathcal{C})$, $n < \omega$, and $p \in \mathbb{P}$. Pick distinct $k, \ell \geq \max\{n, \text{dom}(p)\}$ such that $k, \ell \in \text{dom}(x)$ and $k < \ell$ where $x(k)$ and $x(\ell)$ are not in $\text{ran}(p)$. Choose a $q \in \mathbb{P}$ such that $q \supseteq p$ and the k -th and the ℓ -th unrepeated elements are equal to $x(k)$ and not equal to $x(\ell)$, respectively. Rigorously, let $s = \text{dom}(p)$, $t = |\text{ran}(p)|$, pick distinct $a_0, a_1, \dots, a_{k-t-1}, b_0, b_1, \dots, b_{\ell-k-1} \in \omega \setminus (\text{ran}(p) \cup \{x(k), x(\ell)\})$, and define $q = p \cup \{(s+i, a_i) : i < k-t\} \cup \{(s-t+k, x(k))\} \cup \{(s-t+k+1+j, b_j) : j < \ell-k\}$. Thus $\hat{q}(k) = x(k)$ and $\hat{q}(\ell) \neq x(\ell)$, so $q \in D_{x,n}$. Let

$$\mathcal{D} = \{[D_{x,n}] : x \in \text{bc}(\mathcal{C}) \text{ and } n < \omega\} \cup \{[A_n] : n < \omega\}$$

Then $\mathcal{D}$ is a family of open dense subsets of the Baire space ${}^\omega\omega$ where $|\mathcal{D}| \leq |\mathcal{C}| < \text{cov}(\mathcal{M})$. By the above fact, $\bigcap \mathcal{D} \neq \emptyset$, and we can pick an element $f \in \bigcap \mathcal{D}$. Thus $x \cap \hat{f}$ and $x \setminus \hat{f}$ are infinite where $\hat{f} \in \text{Sym}(\omega)$. So $\mathcal{C} \cup \{\hat{f}\}$ is an i.f. of permutations. $\square$

Another way to prove the above theorem is, by using the fact that $\mathfrak{m}_{\text{ctbl}} = \text{cov}(\mathcal{M})$ and showing that $\mathfrak{m}_{\text{ctbl}} \leq \mathfrak{i}_p$ instead. This can be done by consider the countable poset $\text{Fn}_{1-1}(\omega, \omega)$. We leave the details for the reader.

By simplifying the proof of Theorem 3.1, we can show that $\text{cov}(\mathcal{M}) \leq \mathfrak{i}_f$. However, the following theorem gives a better lower bound of $\mathfrak{i}_f$. Recall that the cardinal $\mathfrak{d}$ is the *dominating number*, the smallest size of a dominating family of functions on ω .

Theorem 3.2. $\mathfrak{d} \leq \mathfrak{i}_f$.

Proof. Suppose $\mathcal{I} \subseteq {}^\omega\omega$ is an independent family with $\aleph_1 \leq |\mathcal{I}| < \mathfrak{d}$. We shall show that $\mathcal{I}$ is not maximal.

Take a model M of sufficiently large finite fragment of ZFC with $\mathcal{I} \in M$ and $|M| = |\mathcal{I}|$.

Claim. There is a strictly increasing sequence $\{n_k : k < \omega\} \subseteq \omega$ with $n_0 = 0$ so that for any $g \in M \cap {}^\omega\omega$, there are infinitely many k such that $g(n_k) < n_{k+1}$.

Proof. Since $|M| < \mathfrak{d}$, ${}^\omega\omega \cap M$ is not a dominating family. Hence there is a strictly increasing function $f \in {}^\omega\omega$ such that $\exists^\infty n [g(n) < f(n)]$ for all $g \in {}^\omega\omega \cap M$. Define $n_0 = 0$ and $n_{k+1} = f(n_k)$ for each $k < \omega$.

Let $g \in {}^\omega\omega \cap M$. We shall show that $\exists^\infty k [g(n_k) < n_{k+1}]$. In M , define $G \in {}^\omega\omega \cap M$ by $G(0) = 1$ and

$$G(n+1) = \max(\{g(i) : i \leq G(n)\} \cup \{G(n)\}) + 1.$$

If there is an $\ell < \omega$ such that $|\text{ran}(G) \cap [n_k, n_{k+1})| \leq 1$ for all $k \geq \ell$, then $G(k) \geq n_{k+1} = f(n_k) \geq f(k)$ for all $k \geq n_\ell + 1$, which is impossible by the property of f . So there are infinitely many k such that $|\text{ran}(G) \cap [n_k, n_{k+1})| \geq 2$. For such a k , there is an a_k such that $n_k \leq G(a_k) < G(a_k + 1) < n_{k+1}$ and hence, by the definition of G , $g(n_k) \leq G(a_k + 1) < n_{k+1}$, and the proof of the claim is done.

Let $\{f_k : k < \omega\} \subseteq \mathcal{I}$ be a sequence in M without repetitions. Define

$$h = \bigcup_{k < \omega} f_k \upharpoonright [n_k, n_{k+1}).$$

We shall show that $\mathcal{I} \cup \{h\}$ is an independent family and $h \notin \mathcal{I}$. To show this, let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{I}$ be disjoint finite sets. Note that h and f_k agree on $[n_k, n_{k+1})$ for each k . It suffices to show that

$$\exists^\infty k < \omega \exists a \in [n_k, n_{k+1}) [\forall f \in \mathcal{A}[f(a) = f_k(a)] \wedge \forall g \in \mathcal{B}[g(a) \neq f_k(a)]] .$$

Choose an $\ell < \omega$ so that $f_k \notin \mathcal{A} \cup \mathcal{B}$ for all $k > \ell$. Since, for any n with $\ell < n < \omega$, three sets $\mathcal{A}$, $\mathcal{B}$ and $\{f_k : \ell < k \leq n\}$ are disjoint subset of an independent family $\mathcal{I}$, working in M , we can construct a $d \in {}^\omega \omega \cap M$ such that for any n, k with $\ell < k \leq n$,

$$\exists a \in [n, d(n)) [\forall f \in \mathcal{A}[f(a) = f_k(a)] \wedge \forall g \in \mathcal{B}[g(a) \neq f_k(a)]] .$$

Since there are infinitely many k such that $d(n_k) < n_{k+1}$ (by the above claim), we are done. $\square$

4. Summary and open problems

We summarise relationships among the cardinals studied in this paper and other well-known ones in the following diagram. A line connecting two cardinals indicates that the lower cardinal is less than or equal to the upper one.

By Cohen forcing, we have that $\aleph_1 = \text{non}(\mathcal{M}) < \text{cov}(\mathcal{M}) = \mathfrak{c}$ is relatively consistent with ZFC (cf. [3, Section 11.3, pages 472–473]). Therefore, the following statement is consistent with ZFC:

$$\aleph_1 = \mathfrak{p} = \mathfrak{s} = \mathfrak{s}_f = \mathfrak{s}_p = \text{non}(\mathcal{M}) < \text{cov}(\mathcal{M}) = \mathfrak{r} = \mathfrak{r}_f = \mathfrak{r}_p = \mathfrak{i}_f = \mathfrak{i}_p = \mathfrak{i} = \mathfrak{c}.$$

By Random forcing, we have that $\aleph_1 = \mathfrak{s} = \text{cov}(\mathcal{M}) < \text{non}(\mathcal{M}) = \mathfrak{r} = \mathfrak{c}$ is relatively consistent with ZFC (cf. [3, Section 11.4, pages 473–474]). Thus it is relatively consistent with ZFC that

$$\aleph_1 = \mathfrak{p} = \mathfrak{r}_f = \mathfrak{s} = \text{cov}(\mathcal{M}) < \text{non}(\mathcal{M}) = \mathfrak{r} = \mathfrak{s}_f = \mathfrak{s}_p = \mathfrak{a}_e = \mathfrak{a}_p = \mathfrak{i} = \mathfrak{c}.$$

Since there are many models of ZFC in which $\text{cov}(\mathcal{M}) = \aleph_1$ and $\mathfrak{d} = \aleph_2$, e.g., Laver, Mathias, or Miller forcing (cf. [3, Sections 11.7–11.9, pages 478–479]), by Theorem 3.2, $\text{cov}(\mathcal{M}) < \mathfrak{i}_f$ in these models.

From the above results, there are some interesting open problems below.

1. Is $\mathfrak{r}_p = \text{cov}(\mathcal{M})$?
2. Is $\mathfrak{d}$ a lower bound of $\mathfrak{i}_p$?
3. Is there any model of ZFC in which $\text{cov}(\mathcal{M}) < \mathfrak{i}_p$?

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References

- [1] T. Bartoszyński, Combinatorial aspects of measure and category, *Fundamenta Mathematicae*, **127** (1987), 225–239.
- [2] M. G. Bell, On the combinatorial principle $P(c)$, *Fundamenta Mathematicae*, **114** (1981), 149–157.
- [3] A. Blass, Combinatorial cardinal characteristics of the continuum, In: *Handbook of Set Theory*, vol. 1, M. Foreman, A. Kanamori (eds.), (Springer, Berlin, 2010), 395–490.

- [4] J. Brendle, O. Spinas, Y. Zhang, Uniformity of the meager ideal and maximal cofinality groups, *Journal of Algebra*, **232** (2000), 209–225.
- [5] D. H. Fremlin, S. Shelah, On partitions of the real line, *Israel Journal of Mathematics*, **32** (1979), 299–304.
- [6] L. J. Halbeisen, Combinatorial set theory: With a gentle introduction to forcing, 2 ed., Springer International Publishing (2017).
- [7] K. Kunen, Set theory, vol. 34 of Studies in logic, College Publications (2011).
- [8] N. Sonpanow, P. Vejjajiva, Independent families of functions and permutations, *Mathematical Logic Quarterly*, **66** (2020), 311–315.
- [9] Y. Zhang, On a class of m.a.d. families, *Journal of Symbolic Logic*, **64** (1999), 737–746.

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