UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA
doi: 10.4467/20843828AM.13.003.2279 FASCICULUS LI, 2013

FINITELY BASED MONOIDS OBTAINED FROM NON-FINITELY BASED SEMIGROUPS

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Abstract. Presently, no example of non-finitely based finite semigroup S is known for which the monoid S^1 is finitely based. Based on a general result of M. V. Volkov, two methods are established from which examples of such semigroups can be constructed.

1. Introduction. A semigroup is *finitely based* if the identities it satisfies are finitely axiomatizable. Commutative semigroups [**9**], idempotent semigroups [**3–5**], and finite groups [**8**] are finitely based, but not all semigroups are finitely based in general. Further, the class $\mathfrak{FB}$ of finitely based semigroups is not closed under common operators such as the formation of homomorphic images, subsemigroups, and direct products. Refer to the survey by Volkov [**14**] for more information on these operators and the finite basis problem for semigroups in general. The present article is concerned with the operator that maps each semigroup S to the smallest containing monoid

$$S^1 = \begin{cases} S & \text{if } S \text{ is a monoid,} \\ S \cup \{1\} & \text{otherwise.} \end{cases}$$

The class $\mathfrak{FB}$ is not closed under this operator; there exist finitely based semigroups S such that the monoids S^1 are non-finitely based. The earliest example demonstrating this property, published by Perkins [**9**] in 1969, is a certain semigroup R_{24} of order 24; see Section 3. Perkins's work in fact contains a much smaller example that he was unaware of at that time: he proved that the Brandt monoid B_2^1 is non-finitely based [**9**], while the Brandt semigroup

$$B_2 = \left\langle a, b \,\middle|\, a^2 = b^2 = 0,\, aba = a,\, bab = b \right\rangle$$

2000 *Mathematics Subject Classification.* 20M05.
Key words and phrases. Semigroup, monoid, finitely based.

of order five was later shown by Trahtman to be finitely based [**12**]. These examples led Shneerson [**11**] to question the existence of semigroups having the "opposite" property.

QUESTION 1. Do non-finitely based semigroups S exist for which the monoids S^1 are finitely based?

In what follows, it is convenient to call a semigroup S *conformable* if S is non-finitely based while S^1 is finitely based. Shneerson provided an affirmative answer to Question 1 by proving that the semigroup

$$T = \langle a, b \,|\, aba = ba \rangle$$

is conformable [**11**]. However, unlike the finite examples B_2 and R_{24} that motivated Shneerson's question, the semigroup T is infinite. Apart from T, no other semigroup has since been found to be conformable. Therefore, the restriction of Question 1 to finite semigroups is of fundamental interest.

QUESTION 2. Do finite conformable semigroups exist?

Recall that a semigroup S with zero 0 is *nilpotent* if there exists some $n \geq 1$ such that the product of any n elements of S equals 0. Each nilpotent semigroup satisfies the identity

$$x_1 x_2 \cdots x_n = y_1 y_2 \cdots y_n$$

for some $n \geq 1$ and so is easily shown to be finitely based [**9**]. It turns out that by the following general result of Volkov [**13**], which was established prior to Question 1 being posed by Shneerson, an abundance of finite conformable semigroups can be constructed from nilpotent semigroups.

LEMMA 3. *Suppose that N is any nilpotent semigroup. Then for any semigroup S, the direct product $S \times N$ is finitely based if and only if S is finitely based.*

2. Constructing finite conformable semigroups. Recall that the *variety* generated by a semigroup S, denoted by $\mathsf{var}\, S$, is the smallest class of semigroups containing S that is closed under the formation of homomorphic images, subsemigroups, and arbitrary direct products. A semigroup S satisfies the same identities as the variety $\mathsf{var}\, S$ it generates [**2**].

THEOREM 4. *Suppose that S and N are any semigroups such that*
(a) *S^1 is non-finitely based*;
(b) *N is nilpotent*;
(c) *$S^1 \times N^1$ is finitely based.*

Then the direct product $P = S^1 \times N$ is conformable.

PROOF. The semigroup P is non-finitely based by (a), (b), and Lemma 3. Since P is a subsemigroup of $S^1 \times N^1$, it belongs to the variety $\mathsf{var}\,(S^1 \times N^1)$. The inclusion $\mathsf{var}\,P^1 \subseteq \mathsf{var}\,(S^1 \times N^1)$ then follows [**1**, Lemma 7.1.1]. But the monoids S^1 and N^1 are embeddable in P^1 so that $\mathsf{var}\,P^1 = \mathsf{var}\,(S^1 \times N^1)$. Therefore, the monoid P^1 is finitely based by (c). □

THEOREM 5. *Suppose that S and N are any semigroups such that*

(a) *S^1 is non-finitely based*;
(b) *N is nilpotent*;
(c) *N^1 is finitely based*;
(d) $\mathsf{var}\,S^1 \subseteq \mathsf{var}\,N^1$.

Then the direct product $P = S^1 \times N$ is conformable.

PROOF. Following the proof of Theorem 4, the semigroup P is non-finitely based with $\mathsf{var}\,P^1 = \mathsf{var}\,(S^1 \times N^1)$. Then (d) implies that $\mathsf{var}\,P^1 = \mathsf{var}\,N^1$, whence the monoid P^1 is finitely based by (c). □

The following results of Jackson and Sapir [**6**] now provide the appropriate finite semigroups S and N to construct the conformable semigroups P in Theorems 4 and 5.

LEMMA 6. *There exist finite nilpotent semigroups S and N such that S^1 and N^1 are non-finitely based while $S^1 \times N^1$ is finitely based.*

LEMMA 7. *There exist finite nilpotent semigroups S and N such that S^1 is non-finitely based, N^1 is finitely based, and* $\mathsf{var}\,S^1 \subseteq \mathsf{var}\,N^1$.

Jackson and Sapir in fact presented methods for locating as many of the semigroups in Lemmas 6 and 7 as desired [**6**, Corollaries 3.1 and 5.2].

3. Explicit examples of finite conformable semigroups. Let $\mathcal{A}^+$ denote the free semigroup over a countably infinite alphabet $\mathcal{A}$. Elements of $\mathcal{A}^+$ are called *words*. For any finite set $\mathcal{W} = \{w_1, \ldots, w_k\}$ of words, let $\mathbf{R}(w_1, \ldots, w_k)$ denote the Rees quotient of $\mathcal{A}^+$ over the ideal of all words that are not factors of any word in $\mathcal{W}$. Equivalently, $\mathbf{R}(w_1, \ldots, w_k)$ can be treated as the semigroup that consists of every nonempty factor of every word in $\mathcal{W}$, together with a zero element 0, with binary operation $\cdot$ given by

$$u \cdot v = \begin{cases} uv & \text{if } uv \text{ is a factor of some word in } \mathcal{W}, \\ 0 & \text{otherwise.} \end{cases}$$

It is easily seen that the semigroup $\mathbf{R}(w_1, \ldots, w_k)$ is nilpotent. The semigroup R_{24} of Perkins introduced in Section 1 is $\mathbf{R}(xyzyx, xzyxy, xyxy, xxz)$.

Consider the semigroups

$$R_8 = \mathbf{R}(xyxy), \quad R_{12} = \mathbf{R}(xxyy, xyyx), \quad \text{and} \quad R_{15} = \mathbf{R}(xyxy, xxyy, xyyx)$$

where $|R_8| = 8$, $|R_{12}| = 12$, and $|R_{15}| = 15$. Then

- R_8^1 is non-finitely based [**6**, Example 4.2];
- R_{12}^1 is non-finitely based [**6**, proof of Corollary 5.1];
- R_{15}^1 is finitely based [**6**, Corollary 3.2 and proof of Corollary 5.1];
- $\mathsf{var}\left(R_8^1 \times R_{12}^1\right) = \mathsf{var}\, R_{15}^1$ [**6**, Lemma 5.1].

It follows that the pairs $(S, N) = (R_8, R_{12})$ and $(S, N) = (R_8, R_{15})$ satisfy Lemmas 6 and 7, respectively. Therefore, by Theorems 4 and 5, the semigroups $R_8^1 \times R_{12}$ and $R_8^1 \times R_{15}$ are conformable.

Now since the conformable semigroup $P = S^1 \times N$ in Theorems 4 and 5 is a direct product, its order $|S^1||N|$ can be quite large in general. But it turns out that the semigroup P contains a proper subsemigroup that is also conformable. Define

$$P_* = S_*^1 \cup N_*$$

where $S_*^1 = \{(a, 0) \mid a \in S^1\}$ and $N_* = \{(0, b) \mid b \in N\}$. Then it is easily seen that S_*^1, N_*, and P_* are subsemigroups of P.

Proposition 8. *The semigroup P_* is conformable.*

Proof. The isomorphic relations $S^1 \cong S_*^1$ and $N \cong N_*$ clearly hold. Therefore,

$$\mathsf{var}\, P = \mathsf{var}\,(S^1 \times N) = \mathsf{var}\,(S_*^1 \times N_*) \subseteq \mathsf{var}\, P_* \subseteq \mathsf{var}\, P,$$

whence the semigroups P and P_* generate the same variety and so satisfy the same identities. The result thus follows. □

The semigroup P_* has order $|S^1| + |N| - 1$ and so is often much smaller than the semigroup P with order $|S^1||N|$. For instance,

$$(|P_*|, |P|) = \begin{cases} (20, 108) & \text{if } (S, N) = (R_8, R_{12}), \\ (23, 135) & \text{if } (S, N) = (R_8, R_{15}). \end{cases}$$

On the other hand, the semigroup P_* is still quite large; the order of any non-finitely based monoid of the form $\mathbf{R}(w_1, \ldots, w_k)^1$ is at least nine [**6**, Theorem 4.3] so that $|P_*| \geq 9 + 2 - 1 = 10$.

In view of the small semigroup B_2 that motivated Question 1, it is natural to pose the following question:

Question 9. What is the smallest possible order of a conformable semigroup?

Based on results of Lee *et al.* [**7**], Sapir [**10**], and Zhang [**15**], the order of any conformable semigroup is at least seven.

Acknowledgment. The author would like to thank the anonymous reviewer for a number of helpful, constructive comments.

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Received September 30, 2013

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