REPORTS ON MATHEMATICAL LOGIC
49 (2014), 79–97
doi:10.4467/20842589RM.14.005.2275

Jerzy MYCKA

RECURSIVELY ENUMERABLE SETS AND WELL-ORDERING OF THEIR ENUMERATIONS

A b s t r a c t. We will introduce the special kind of the order relations into recursively enumerable sets and prove that they can be used to distinguish (albeit in a non-constructive way) between recursive and non-recursive sets.

1. Introduction

Considering sets of natural numbers from the computational point of view we distinguish as the main class of sets the collection of recursively enumerable sets. However, inside this class we can see the crucial difference which lies between recursive and non-recursive sets.

In this paper we use ordinal numbers to indicate recursiveness of sets. We do not employ ordinals in the way which was used to create hierarchies of natural functions (as can be found in the papers [3], [7]), but instead we

Received 4 March 2014

introduce well-order relations according to the method of an enumeration of a set. This gives us a precise criterion, which recognises between recursive and non-recursive sets.

This method can be seen as very natural: if the main characteristic of recursively enumerable sets is given by the fact that they can be listed then the precise level of their computability has to be bound to the degree of the order (or disorder) of their enumeration. The most natural way to measure such kind of complexity would be given by ordinal numbers.

This direction of research is justified by results: we can use ordinal numbers to distinguish recursive and non-recursive recursively enumerable sets. Additionally we can present that many properties of such orderings can be computably (or relatively computably) tested.

The article is written in the self-explanatory way. First we recall fundamental notions of computability and ordinal numbers. In the next section we introduce a special kind of well-order and define an order type for recursively enumerable sets. Then we present some properties of such orders and finally we give the main result, which states that non-recursive recursively enumerable sets have their ordinal (according to the mentioned relation) equal to ω^2.

2. Fundamental notions

Let us start with some useful notation (cf. [1]), which will be used in the following definitions. Let $\mathfrak{F}$ be a class of functions, $\mathcal{F} \subseteq \mathfrak{F}$ be a given subset of functions from $\mathfrak{F}$, and $\mathcal{O} \subseteq \cup_{k\in\mathbb{N}}\{O : \mathfrak{F}^k \to \mathfrak{F}\}$ be a set of operators. The inductive closure $\mathcal{A}$ of $\mathcal{F}$ for $\mathcal{O}$ is the smallest set containing $\mathcal{F}$, such that if $f_1, \ldots, f_k \in \mathcal{A}$ are in the domain of the k-ary $O \in \mathcal{O}$, then $O(f_1, \ldots, f_k) \in \mathcal{A}$. When presented together with $\mathcal{O}$, the inductive closure $\mathcal{A} = [\mathcal{F}; \mathcal{O}]$ is called a function algebra. Usually we write members of $\mathcal{F}$ and $\mathcal{O}$ not enclosed by parenthesis, but these two sets will be separated in definitions by a semicolon.

Rudiments of theory of computable functions and sets

Let us consider the class $\mathfrak{F}$ of partial functions over $\mathbb{N}^k, k \geq 1$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. Important examples of functions in $\mathfrak{F}$ are: the zero function z, $z(x) = 0$; the successor function s given by $s(x) = x + 1$; and the set of projection functions u_i^n, where for $1 \leq i \leq n$ we have $u_i^n(x_1, \ldots, x_n) = x_i$. From this moment we will write $\vec{x}$ to designate an arbitrary sequence $\vec{x} = x_1, \ldots, x_n$.

We consider the composition operators c^m, such that for every $g : \mathbb{N}^m \to \mathbb{N}, h_1, \ldots, h_m : \mathbb{N}^n \to \mathbb{N}$, the function $\mathsf{c}^m(g, h_1, \ldots, h_m) : \mathbb{N}^n \to \mathbb{N}$ is given by

$$\mathsf{c}^m(g, h_1, \ldots, h_m)(\vec{x}) = g(h_1(\vec{x}), \ldots, h_m(\vec{x})).$$

We also use the primitive recursion operator p, which for every given $g : \mathbb{N}^n \to \mathbb{N}$ and $h : \mathbb{N}^{n+2} \to \mathbb{N}$, sets

$$\mathsf{p}(g, h)(\vec{x}, 0) = g(\vec{x}) \text{ and } \mathsf{p}(g, h)(\vec{x}, y + 1) = h(\vec{x}, y, \mathsf{p}(g, h)(\vec{x}, y)).$$

Definition 2.1. The class PRIM of primitive recursive functions is given by the function algebra

$$\text{PRIM} = [z, s, u_i^n; \mathsf{c}^m, \mathsf{p}].$$

We can also introduce the operator μ of unbounded minimalisation defined in the following way: for any function $f : \mathbb{N}^{k+1} \to \mathbb{N}$ in $\mathfrak{F}$ we can find the new function $\mu_y(f)$ given as below:

$$\mu_y(f)(\vec{x}) =$$
$$= \min\{y : f(\vec{x}, y) = 0 \text{ and } (\forall z < y) f(\vec{x}, z) \text{ is defined and not equal to } 0\}.$$

Let us indicate that this operator is the origin of partiality (i.e. a property of being not everywhere defined) of partial recursive functions introduced in the following definition.

Definition 2.2. The class PREC of partial recursive functions is given by the following function algebra

$$\text{PREC} = [z, s, u_i^n; \mathsf{c}, \mathsf{p}, \mu].$$

We can restrict the class PREC by imposition of the additional condition of totality of its members (i.e. all functions should be everywhere defined). In this case we obtain the set REC of (total) recursive functions.

Sometimes we would like to use wider sets of functions admitting recognition of members of some (freely chosen) set $A \subseteq \mathbb{N}$, i.e. adding the characteristic function K_A of this set A to basic functions.

Definition 2.3. The class PREC^A of partial functions is given as follows

$$\mathrm{PREC}^A = [z, s, u_i^n, K_A; \mathsf{c}, \mathsf{p}, \mu].$$

Analogously, REC^A is defined as the subset of total functions from PREC^A.

Functions from PREC^A (REC^A) are called partial A-recursive (respectively A-recursive) functions.

In this paper we are interested in sets rather than functions, so we need some additional ideas from the field of computability theory.

Definition 2.4. A set $A \subseteq \mathbb{N}$ is called a recursive set iff there exists a function $K_A : \mathbb{N} \to \mathbb{N}, K_A \in \mathrm{REC}$ such that

$$K_A(x) = \begin{cases} 1 & x \in A, \\ 0 & x \notin A. \end{cases}$$

A set $A \subseteq \mathbb{N}$ is called a recursively enumerable set iff A is the empty set or there exists a function $f_A : \mathbb{N} \to \mathbb{N}, f_A \in \mathrm{REC}$ such that

$$f_A(\mathbb{N}) = A.$$

Let us add the special notion of an index function for a set A and its function $f_A \in \mathrm{REC}$:

$$\mathrm{index}_{f_A}(x) = \begin{cases} 0 & x \notin A, \\ 1 + \min_i\{i : f_A(i) = x\} & x \in A. \end{cases}$$

It can be observed that for $f_A \in \mathrm{REC}$ the function index_{f_A} is in REC^A.

We will add a few useful results concerning recursive and recursively enumerable sets (the most comprehensive surveys can be found in [4], [6]). First we present a few different characterisations of recursively enumerable sets.

Lemma 2.5. *A set* $A \subseteq \mathbb{N}$ *is recursively enumerable iff* A *is the domain of some partial recursive function* $f : \mathbb{N} \to \mathbb{N}, f \in \mathrm{PREC}$ *iff* A *is the range of some partial recursive function* $g : \mathbb{N} \to \mathbb{N}, g \in \mathrm{PREC}$ *iff* A *is the empty set* $\varnothing$ *or* A *is the finite set or* A *is the range of some one-to-one total recursive function* $h : \mathbb{N} \to \mathbb{N}, h \in REC$.

We should note that in the last case of the above lemma we use one-to-one functions, which is a rarely used but an equivalent modification of the standard definition (cf. [8]).

There are important connections between recursive and recursively enumerable sets. The obvious consequence of Definition 2.4 can be stated simply as the following lemma.

Lemma 2.6. *Every recursive set is recursively enumerable.*

However these two classes are not identical, we can present examples of sets which are recursively enumerable and not recursive; the most typical example is the set $K = \{x \in \mathbb{N} : \phi_x(x) \text{ is defined}\}$, where ϕ_i is a computable enumeration of all one-argument functions from PREC. Another fruitful observation which gives conditions for a recursively enumerable set to be recursive is presented in Kleene's theorem.

Theorem 2.7. *A set* $A \subseteq \mathbb{N}$ *is recursive iff* A *and its complement* $\bar{A}$ *are recursively enumerable.*

Let us hint at another property which also guarantees that infinite recursively enumerable set is recursive.

Lemma 2.8. *Let* A *be infinite recursively enumerable set, then* A *is recursive iff there exists the increasing function* $f_A : \mathbb{N} \to \mathbb{N}$ *such that* $f_A \in \mathrm{REC}$ *and* $f_A(\mathbb{N}) = A$.

Basic facts about ordinal numbers

Let us recall a few basic facts about ordinal numbers. Because ordinal numbers are strongly connected with sets (in fact they are some specific sets), we need to use some fundamental notions of set theory.

We will consider sets as they are described by Zermelo-Fraenkel axioms (see e.g. [2]). We are not interested in axiomatic systems here, so we only informally present what is needed using ideas taken from [5].

Sets are collections of elements, which are themselves sets. So we have to start our constructions with a crucial element of the empty set $\varnothing$. A set y will be called a transitive set iff for every $x \in y$ we have $x \subset y$. This means that any transitive set has as its elements all members of its elements. Now let us introduce a relation of partial order $\leq_y$ on a set y as the relation satisfying for all $x_1, x_2, x_3 \in y$ the following conditions: 1) $x_1 \leq_y x_1$ (reflexivity); 2) $x_1 \leq_y x_2$ and $x_2 \leq_y x_1$ imply $x_1 = x_2$ (antisymmetry); 3) $x_1 \leq_y x_2$ and $x_2 \leq_y x_3$ imply $x_1 \leq_y x_3$ (transitivity). A relation of partial order $\leq_y$ is linear iff every two elements of y are comparable, i.e. for every $x_1, x_2 \in y$ we have $x_1 \leq_y x_2$ or $x_2 \leq_y x_1$. We finally arrive to the most important property - a set y is well-ordered iff y is a linearly ordered set and every subset of y has a minimum, more formally:

$$(\forall z \subseteq y)(\exists x_1 \in z)(\forall x_2 \in z)\ x_1 \leq_y x_2.$$

We are ready to define ordinal numbers (sometimes simply called ordinals).

Definition 2.9. An ordinal number is a transitive set y well-ordered by the relation $\bar{\in}$ defined in the following way:

$$(\forall x_1, x_2 \in y)[x_1 \bar{\in} x_2 \iff (x_1 = x_2) \text{ or } (x_1 \in x_2)].$$

Now we can present examples of ordinal numbers.

Example 2.10. Let us start with the simplest ordinal number $\varnothing$ and call it $\bar{0}$. Now we can construct the finite ordinals $\bar{1} = \{\bar{0}\} = \{\varnothing\}$, $\bar{2} = \{\bar{0}, \bar{1}\} = \{\varnothing, \{\varnothing\}\}$, ..., $\bar{n} = \{\bar{0}, \bar{1}, \dots, \overline{n-1}\} = \{\varnothing, \{\varnothing\}, \dots, \{\varnothing, \{\varnothing\}, \dots \underbrace{\} \dots \}}_{n-1} \}$.

The first infinite ordinal is denoted as $\omega = \{\bar{0}, \dots, \bar{n}, \dots\}$, we can proceed further with some examples of infinite ordinals, e.g. $\{\bar{0}, \dots, \bar{n}, \dots, \omega\}$.

From this moment we will use the first Greek letters to denote ordinal numbers. To obtain more clear picture of ordinals we will add a short explanation about operations defined on ordinal numbers. The very first one is the successor of an ordinal which can be defined as follows: $S(\alpha) = \alpha \cup \{\alpha\}$. This operation can be used to distinguish two kinds of ordinal numbers: α is called a successor ordinal iff there exists an ordinal β such that $\alpha = S(\beta)$; $\alpha \neq \bar{0}$ is called a limit ordinal iff α is not a successor ordinal. We can see that ω is the first limit ordinal.

In the following sections, where it will not lead to any confusion we will identify natural numbers $n \in \mathbb{N}$ with their ordinal counterparts $\bar{n} \in \omega$ and vice versa.

Usually we can define new operations on ordinals using inductive definitions based on the observation that every ordinal number is 0, a successor ordinal $S(\alpha)$ or a limit ordinal β, in the latter case β is the least upper bound of all its predecessors (in this sense ω is the least upper bound of $\bar{0}, \bar{1}, \ldots, \bar{n}, \ldots$), which is equal to the union of all these predecessors. Let us present definitions of this kind for addition, multiplication and exponentiation (γ is a limit ordinal number):

$$\begin{aligned} \alpha + \bar{0} &= \alpha, & \alpha \cdot \bar{0} &= \bar{0}, & \alpha^{\bar{0}} &= \bar{1}, \\ \alpha + S(\beta) &= S(\alpha + \beta), & \alpha \cdot S(\beta) &= \alpha \cdot \beta + \alpha, & \alpha^{S(\beta)} &= \alpha^{\beta} \cdot \alpha, \\ \alpha + \gamma &= \bigcup_{\delta < \gamma} (\alpha + \delta); & \alpha \cdot \gamma &= \bigcup_{\delta < \gamma} (\alpha \cdot \delta); & \alpha^{\gamma} &= \bigcup_{\delta < \gamma} (\alpha^{\delta}). \end{aligned}$$

Using these operations we can construct the next ordinal numbers (still having the countable list of elements). Let us start with some simple examples: $\omega + 1 = \{\bar{0}, \ldots, \bar{n}, \ldots, \omega\}$, $\omega + 2 = \{\bar{0}, \ldots . \bar{n}, \ldots, \omega, \omega + 1\}$ and then we can obtain $\omega + \omega = \omega \cdot 2 = \{\bar{0}, \ldots, \bar{n}, \ldots, \omega, \omega + 1, \ldots, \omega + k, \ldots\}$. Now we can repeat the same process to build $\omega \cdot 3$, $\omega \cdot 4$ and so on to we can reach $\omega \cdot \omega = \omega^2$. We can continue to build higher powers ω^3, ω^4 and even finding ω^{ω}, but still we could proceed with such new ordinals as $\omega^{\omega^{\omega}}, \omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}}$. As the next stage we can use the ordinal number ϵ_0 obtained from the sequence of such towers of powers and having the property $\omega^{\varepsilon_0} = \varepsilon_0$. This is not the end of the road: we can consider next ordinal numbers reaching the first uncountable ordinal ω_1 and finding much more of ordinals in the further (infinite) stages.

However we need only relatively short initial segment of ordinals placed below ϵ_0. For such small ordinal numbers we will use the special notation for their description: namely Cantor normal form. Every non-zero ordinal number $\alpha < \epsilon_0$ can be uniquely written as

$$\omega^{\beta_1} \cdot c_1 + \omega^{\beta_2} \cdot c_2 + \cdots + \omega^{\beta_k} \cdot c_k,$$

where $k, c_1, c_2, \ldots, c_k$ are natural numbers, $\beta_1 > \beta_2 > \ldots > \beta_k \geq 0$ are ordinals and $\beta_1 < \alpha$.

Now we can use ordinals to measure order type of natural sets which are well-ordered.

Definition 2.11. Let $A \subseteq \mathbb{N}$ be a set equipped with some well-ordering relation $\leq_A$. We will call the ordinal α order type of $\langle A, \leq_A \rangle$ iff there exists one-to-one function $f : A \to \alpha$ preserving order i.e such that for any $x, y \in A$

$$x \leq_A y \iff f(x) \bar{\in} f(y).$$

Example 2.12. Let us start with a simple example. We will introduce for the whole set $\mathbb{N}$ the following order

$$x \leq_1 y \iff \begin{cases} x \text{ is an odd number and } y \text{ is an even number,} \\ \text{or } x, y \text{ are both odd and } x \leq y, \\ \text{or } x, y \text{ are both even and } x \leq y. \end{cases}$$

It is simple to observe that $\leq_1$ is a well-order. The set $\mathbb{N}$ could be listed in that order as follows

$$1, 3, \ldots, 2i + 1, \ldots, 0, 2, 4, \ldots, 2j, \ldots.$$

Now let us define the function $f : \mathbb{N} \to \omega \cdot 2$:

$$f(x) = \begin{cases} \frac{x-1}{2} & \text{for } x \text{ odd,} \\ \omega + \frac{x}{2} & \text{for } x \text{ even;} \end{cases}$$

such a function is clearly one-to-one and preserves the order in the sense given in Definition 2.11, so the structure $\langle \mathbb{N}, \leq_1 \rangle$ has the order type $\omega \cdot 2$.

Now let us consider the set $A \subseteq \mathbb{N}$ containing only non-zero powers of prime numbers $A = \{p_i^n : i \in \mathbb{N}, n \in \mathbb{N} - \{0\}\}$, where p_i = i-th prime number counting indexes of primes from zero, i.e. $p_0 = 2, p_1 = 3, p_2 = 5,$ etc. We will equip the set A with the order relation $\leq_2$:

$$p_i^n \leq_2 p_j^m \iff \begin{cases} i < j \text{ (i.e. } p_i < p_j) \\ \text{or } i = j \text{ and } n \leq m. \end{cases}$$

The set A in that order appears as follows

$$\{2, 4, \ldots, 2^i, \ldots, 3, 9, \ldots, 3^k, \ldots, \ldots, p_j, p_j^2, \ldots, p_j^n, \ldots, \ldots\}.$$

We construct the function between A and ω^2 in the following manner

$$g(p_i^n) = \omega \cdot i + (n - 1).$$

We could verify that g is one-to-one function from A to ω^2 such that

$$p_i^n \leq_2 p_j^m \iff g(p_i^n) \bar{\in} g(p_j^n),$$

hence $\langle A, \leq_2 \rangle$ has order type ω^2.

It is important to add that for every well-ordered set $\langle A, \leq_A \rangle$ there exists a one-to-one function from A onto some ordinal α which preserves the order (see [2]). This means that every well-ordered set has an order type.

3. Order in recursively enumerable sets

Let us start with a description how we will introduce order relations into recursively enumerable sets. Let us emphasise that from that point we will restrict our attention only to **infinite** recursively enumerable sets.

Our motivation is based on the simple characterisation of Lemma 2.8: the simplest kind of recursively enumerable sets (i.e. recursive sets) can have all members of such sets computably listed as an increasing sequence. However the more complicated pattern of listing is connected with non-recursive sets.

In this section we will prove computability of many ingredients of well-order inside recursively enumerable sets. Moreover using the following lemmas we will be able to present Corollary 3.11, which is the basis for a separation of recursive and non-recursive recursively enumerable sets by means of ordinal numbers.

The first result is a consequence of Lemma 2.8.

Lemma 3.1. *Let us consider recursively enumerable, non-recursive set* $A \subseteq \mathbb{N}$. *Then for every one-to-one function* $f_A : \mathbb{N} \to \mathbb{N}$ *such that* $f_A(\mathbb{N}) = A, f \in \mathrm{REC}$ *we have:*

$$\neg \exists x_0 \forall (x \geq x_0)[f_A(x+1) > f_A(x)].$$

Proof. We will use *reductio ad absurdum.* Let us assume that there is some function $g_A : \mathbb{N} \to \mathbb{N}$ such that $g_A \in \mathrm{REC}, g_A(\mathbb{N}) = A$ and

$$\exists x_1 \forall (x \geq x_1)[g_A(x+1) > g_A(x)].$$

Then there exists $x_0 \geq x_1$ such that

$$\forall(x \geq x_0)[g_A(x+1) > g_A(x)] \text{ and } \forall(x < x_0)[g(x) < g(x_0)].$$

We can construct the following function $g'_A : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$:

$$g'_A(y,0) = \min\{g_A(z) : z \leq y\};$$
$$g'_A(y,x+1) = \begin{cases} \min\{g_A(z) > g'_A(y,x) : z \leq y\} & x+1 \leq y, \\ g_A(x+1) & x+1 > y. \end{cases}$$

It is clear that g'_A is defined by operations of recursion and bounded minimalisation on recursive functions, so g'_A itself is recursive. According to this definition $g''_A(x) = g'_A(x_0, x)$ is a strictly increasing function as the function of x (where x_0 is taken from our assumption). Moreover $g''_A(\mathbb{N}) = A$ because g'' differs from g_A only by a permutation of finite number of values. Hence A is a recursive set - the contradiction. □

The above observation suggests that difficulty of a recursively enumerable set A is connected with the order of elements in the sequence generated by the function $f_A \in \mathrm{REC}$ such that $f_A(\mathbb{N}) = A$. For recursively enumerable sets which are recursive we have simply increasing sequence, when for non-recursive recursively enumerable sets the pattern is more complicated. Inspired by this fact we can introduce the specific relation for elements of recursively enumerable sets.

Definition 3.2. Let A be an infinite recursively enumerable set and $f_A \in \mathrm{REC}$ satisfies $f_A(\mathbb{N}) = A$, f_A is one-to-one. Then we can define levels of A (with respect to f_A) in the following way:

$$L^0_{f_A} = \{x^0_0, \dots, x^0_i, \dots\},$$

where its members are given as follows

$$x^0_0 = f_A(0),$$
$$x^0_{i+1} = \min\{y \in A : y > x^0_i \text{ and } \mathrm{index}_{f_A}(y) > \mathrm{index}_{f_A}(x^0_i)\};$$

and the higher levels are defined recursively

$$L^{j+1}_{f_A} = \{x^{j+1}_0, \dots, x^{j+1}_i, \dots\},$$

where its members are given analogously

$$x_0^{j+1} = \min\{y \in A : y \notin \bigcup_{k=0}^{j} L_{f_A}^k\},$$
$$x_{i+1}^{j+1} = \min\{y \in A : y \notin \bigcup_{k=0}^{j} L_{f_A}^k \text{ and } y > x_i^{j+1} \text{ and } \mathrm{index}_{f_A}(y) > \mathrm{index}_{f_A}(x_i^{j+1})\}.$$

Such a construction does not need to have infinitely many levels. It is possible to have $L_{f_A}^j = \varnothing$ for all j greater then some given $k \in \mathbb{N}$. It is also possible that on the last non-empty level the number of elements is finite.

Let us prove that this construction of levels is sufficient to exhaust all elements of the recursively enumerable set A.

Lemma 3.3. *Let $A \subseteq \mathbb{N}$ be a recursively enumerable set and $f_A \in \mathrm{REC}$ a one-to-one function that satisfies $f_A(\mathbb{N}) = A$. Then*

$$A = \bigcup_{i \in \omega} L_{f_A}^i.$$

Proof. It is sufficient to observe that every $x \in A$ is equal to $f_A(y)$ for some $y \in \mathbb{N}$ and we cannot start more than $y + 1$ levels on our way through the segment $(f_A(0), \dots, f_A(y))$. Let us use the auxiliary sequence of levels defined as $M_{f_A}^i = \bigcup_{j \in \{0,\dots,i\}} L_{f_A}^j$, then our x must belong to $M_{f_A}^y$.

In this way for every $y \in \mathbb{N}$ we obtain

$$\{f_A(0), \dots, f_A(y)\} \subseteq M_{f_A}^y = \bigcup_{j \in \{0,\dots,y\}} L_{f_A}^j.$$

Taking unions of both sides for all $y \in \omega$ we obtain

$$A \subseteq \bigcup_{j \in \omega} L_{f_A}^j.$$

Since it is obvious from the definition of $L_{f_A}^j$ that every $L_{f_A}^j \subseteq A$, we finally have $A = \bigcup_{i \in \omega} L_{f_A}^i$. □

We will analyse the basic computational properties of such orders.

Lemma 3.4. *Let A be recursively enumerable set such that $A = f_A(\mathbb{N})$, where f_A is a one-to-one recursive function, than the function*

$$K_{f_A}(i, x) = \begin{cases} 1 & x \in L_{f_A}^i, \\ 0 & x \notin L_{f_A}^i \end{cases}$$

is A-recursive.

Proof. Let us observe that the first test involves checking whether $x \in A$, which is obviously done by the A-recursive function K_A. Then in positive case ($x \in A$) we need to check successively $x \in L^0_{f_A}, \ldots, x \in L^i_{f_A}$. But every such process is clearly computable. First we need to compare x with the increasing sequence of elements x^0_k from $L^0_{f_A}$ only to the first moment when $x^0_k \geq x$. But elements x^0_k of $L^0_{f_A}$ can be computably generated by an enumeration of increasing values from $f_A(0), f_A(1), \ldots$. If $x^0_k = x$ then the answer is negative, otherwise, for some k, we have $x^0_k > x$ and we start the second stage of the test.

In the same manner we compare x with the next elements x^1_k taken from $L^1_{f_A}$. For this purpose we restart our listing of $f_A(0), f_A(1), \ldots$ but this time we remember which elements were marked as belonging to $L^0_{f_A}$. Now we start with the first element $f_A(k_1)$ which is smaller than its predecessor $f_A(k_1-1)$ and we construct the increasing sequence from generated elements of $f(A)$, which does not belong to the initial part of $L^0_{f_A}$. We will proceed only to the moment where $x \geq x^1_k$ (in necessary cases we have to enhance the computed initial segment of $L^0_{f_A}$ but always only about some finite sequence of values).

We deal analogously with the next sequences taken from $L^2_{f_A}, \ldots L^{i-1}_{f_A}$. If in all these cases the answer is negative then we compare x with the sequence taken from $L^i_{f_A}$ to the moment when $x \geq x^i_k$ for some k. If we have $x = x^i_k$ then the final answer is positive, otherwise the answer is negative one.

We will describe this process more formally. Let us create - adding them successively on next stages - sets $S^i_{f_A}$ and start them all empty except $S^0_{f_A} = \{f_A(0)\}$. Now for any element $f_A(m)$ do the following test: if $f_A(m) > \max S^0_{f_A}$ then $f_A(m)$ is in $L^0_{f_A}$ and modify $S^0_{f_A} = S^0_{f_A} \cup \{f_A(m)\}$, if not then check $f_A(m) > \max S^1_{f_A}$ (we take $\max \varnothing = -1$) and in the positive case do $S^1_{f_A} = S^1_{f_A} \cup \{f_A(m)\}$ otherwise continue for $S^2_{f_A}, \ldots, S^m_{f_A}$. The element $f_A(m)$ has to be added to one of these finite sets. In this way we can have the initial segment of any $L^i_{f_A}$ of any needed finite length. Simultaneously checking $f_A(m) = x$ we are able to find for any $x \in A$ its level.

It is important to observe that we can generate recursively elements from $L^k_{f_A}$ by choosing increasing sequences from the sequence $f_A(0), \ldots, f_A(k), \ldots$. Hence checks for x on the different levels $L^i_{f_A}$ are recursive, because always executed only finite number of times.

Because all described operations can be translated into appropriate recursive functions and the first step is A-recursive so the function K_{f_A} is A-recursive too.

Let us observe that if we want to check whether $x \in A$ belongs to some non- existent level $L^j_{f_A}$ then this element would be found on the earlier stage of the construction and we would obtain the negative (i.e. correct) answer. □

For future use let us add some modification of K_{f_A}, namely

$$K^*_{f_A}(i,x) = \begin{cases} 1 & x \in L^j_{f_A} \text{ for any } j \le i, \\ 0 & \text{otherwise;} \end{cases}$$

such function can be defined by the operation of simple recursion on i from the function K_{f_A}:

$$\begin{aligned} K^*_{f_A}(0,x) &= K_{f_A}(0,x), \\ K^*_{f_A}(i+1,x) &= K^*_{f_A}(i,x) + K_{f_A}(i+1,x). \end{aligned}$$

From this description we obtain the following corollary.

Corollary 3.5. *The function $K^*_{f_A}$ is A-recursive.*

Now we are ready to give the definition of the mentioned above relation.

Definition 3.6. Let $A \subseteq \mathbb{N}$ be an infinite recursively enumerable set with a one-to-one recursive function f_A such that $f_A(\mathbb{N}) = A$. Then we will define the relation $\le_{f_A} \subseteq A \times A$ for $x \in L^i_{f_A} \subseteq A$, $y \in L^j_{f_A} \subseteq A$ in the following manner

$$x \le_{f_A} y \iff (i < j) \text{ or } (i = j \text{ and } x \le y).$$

To confirm that $\le_{f_A}$ is a partial order it is enough to substitute the relation $\le_{f_A}$ into respective conditions for reflexivity, antisymmetry, transitivity and check their obvious validity. It is also quite clear that every two elements of A are comparable by $\le_{f_A}$: they are either on different levels $L^i_{f_A}, L^j_{f_A}, i \ne j$ or they are on the same level and can be compared through the standard relation $\le$. Moreover, we can prove the minimum property for every subset of A. First let us observe that every subset $B \subseteq A$ can be divided into its levels $L^i_{B,f_A} = B \cap L^i_{f_A}$ for $i \in \mathbb{N}$. Some of these levels can be empty but the non-empty levels are ordered by the standard well-ordered relation $\langle \mathbb{N}, \le \rangle$ on their indexes. So we can find the level $L^{i_0}_{B,f_A}$ with the minimal index given by i_0 and every element of $L^{i_0}_{B,f_A}$ is earlier (by the definition of $\leq_{f_A}$) than any element from the rest of levels L^{j}_{B,f_A}. Inside the level $L^{i_0}_{B,f_A}$ all elements are ordered by the usual relation $\leq$ (accordingly to the definition of $\leq_{f_A}$) and consequently we can find the minimal element in this subset and, moreover, this element is minimal for the whole set B. These remarks give us the following consequence.

Theorem 3.7. *The relation $\leq_{f_A}$ for recursively enumerable set $A \subseteq \mathbb{N}$ with a one-to-one recursive function f_A such that $f_A(\mathbb{N}) = A$ is a relation of well-order.*

Hence, using the above theorem and the mentioned property that every well-ordered set has its order type we can obtain the fundamental corollary.

Corollary 3.8. *Every infinite recursively enumerable set A with the order $\leq_{f_A}$ induced by a one-to-one recursive function f_A such that $f_A(\mathbb{N}) = A$ has order type.*

Definition 3.2 is constructed by using the levels $L^i_{f_A}$ of the set A. It would be helpful to determine in what sense we can compute some indexes of given element.

Lemma 3.9. *Let us denote by $l_{f_A}, i_{f_A} : \mathbb{N} \to \mathbb{N}$ and $v_{f_A} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such functions that*

$$l_{f_A}(x) = \begin{cases} j+1 & x \in L^j_{f_A}, \\ 0 & x \notin A; \end{cases}$$

$$v_{f_A}(i,j) = \begin{cases} x & x \in L^i_{f_A} \textit{ and} \\ & \textit{there exist exactly } j \textit{ elements } x_0, \ldots, x_{j-1} \\ & \textit{such that } \forall (0 \leq k < j)[x_k \in L^i_{f_A} \textit{ and } x_k < x], \\ \textit{undefined} & \textit{otherwise}; \end{cases}$$

$$i_{f_A}(x) = \begin{cases} 0 & x \notin A, \\ j+1 & x \in L^i_{f_A} \subseteq A \textit{ and } v_{f_A}(i,j) = x. \end{cases}$$

Then the functions l_{f_A}, i_{f_A} are A-recursive, v_{f_A} is partial A-recursive function, and, moreover, the order $\leq_{f_A}$ (precisely: the characteristic function of this relation) is A-recursive.

Proof. It is sufficient to use the above defined function K_{f_A} and the characteristic function K_A of the set A to obtain:

$$l_{f_A}(x) = \begin{cases} 0 & K_A(x) = 0, \\ 1 + \mu_y[K_{f_A}(y,x) = 1] & \text{otherwise.} \end{cases}$$

Because the μ-operation is total in this context (if $x \in A$ then there must be some level containing x) this definition uses only recursive and A-recursive (K_A, K_{f_A}) components, so l_{f_A} is A-recursive too.

Now we will use the above result to obtain a straightforward consequence about $\leq_{f_A}$. We will rewrite Definition 3.6 using A-recursive (or recursive) functions in the following way: This can be done in the following way:

$$K_{\leq_{f_A}}(x,y) = \begin{cases} 1 & x \leq_{f_A} y \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & (l_{f_A}(x) < l_{f_A}(y)) \vee (l_{f_A}(x) = l_{f_A}(y) \wedge x \leq y), \\ 0 & \text{otherwise.} \end{cases}$$

The above expression can be simply transformed into more formal (and less readable) form built from A-recursive and recursive functions, which guarantees that $\leq_{f_A}$ is A-recursive.

In the next step let us indicate that $v_{f_A}(i,j)$ gives j-th element from the increasing sequence built on the level $L^i_{f_A}$. The first step in a construction of v_{f_A} is an analysis where we can find the smallest elements $v'_{f_A}(i)$ of the consecutive levels $L^i_{f_A}$. Of course $v'_{f_A}(0) = f_A(0)$; now we can find the smallest element on the level $L^{i+1}_{f_A}$ as the first not included in the previous levels $L^0_{f_A}, \ldots, L^i_{f_A}$, hence

$$v'_{f_A}(i+1) = f_A(\mu_y[f_A(y) \notin L^0_{f_A} \cup \ldots \cup L^i_{f_A}]) = f_A(\mu_y[K^*_{f_A}(i, f_A(y)) = 0]).$$

In this way we have defined the partial A-recursive function v'_{f_A} - its partiality is due to possibility that the set A with regard to $\leq_{f_A}$ could have only finite number of k non-empty levels, A-recursiveness is implied by only recursive and A-recursive functions used in this definition done by means of the μ-operation.

Having found the first elements on the all existing levels we can proceed with the further elements on the same levels by a simple enumeration:

$$\begin{aligned} v_{f_A}(i,0) &= v'_{f_A}(i), \\ v_{f_A}(i,j+1) &= \mu_y[y \in L^i_{f_A} \text{ and } y > v_{f_A}(i,j)] \\ &= \mu_y[K_{f_A}(i,y) = 1 \text{ and } y > v_{f_A}(i,j)]. \end{aligned}$$

Once again we should note a possibility of partiality: either some level $L^i_{f_A}$ does not exists, then $v_{f_A}(i,0)$ and consequently all $v_{f_A}(i,j)$ for $j > 0$ are undefined or the last level has only finite number of members, then minimalisation become undefined after finite number of steps. So, the above method gives the function v_{f_A} as a partial A-recursive function.

The next useful function $i_{f_A}(x)$ gives the index of $x \in A$ on its proper level. Hence $i_{f_A}(x)$ informs us how far is x from the beginning of its level $l_{f_A}(x) \neq 0$. We can describe the computation of i_{f_A} in the following way which can be simply coded as a formal recursive definition. First we will check by K_A whether x is in A, if not the answer is 0; otherwise we will find the smallest i such that $x \in L^i_{f_A}$ and later we will find the smallest j such that $v_{f_A}(i,j) = x$. □

With these functions we can define (partial) functions and (total) predicates which describe properties of elements of A with respect to the order $\leq_A$.

Lemma 3.10. *Let $s_{f_A} : \mathbb{N} \to \mathbb{N}$ be a partial function such that $s_{f_A}(x)$ is the immediate successor of x with respect to the order $\leq_A$; $li_{f_A} : \mathbb{N} \to \mathbb{N}$ be a partial function such that $li_{f_A}(x)$ is the next limit number after x with respect to $\leq_{f_A}$. Let $K_{s_{f_A}}(x,y)$ be a (total) characteristic function of the relation 'y is the immediate successor of x' and $K_{li_{f_A}}(x,y)$ be a (total) characteristic function of the relation 'y is the the first limit number after x' (both with respect to $\leq_{f_A}$). Then s_{f_A}, li_{f_A} are partial A-recursive, $K_{s_{f_A}}, K_{li_{f_A}}$ are A-recursive.*

Proof. Let us start with a construction of $K_{s_{f_A}}(x,y)$. We have to test whether both x, y are in A. If the answer is positive then we will check $l_{f_A}(x) = l_{f_A}(y)$. If indeed x, y are on the same level we have to do that last test $i_{f_A}(x) + 1 = i_{f_A}(y)$. We give the result 1 for $K_{s_{f_A}}(x,y)$ if that condition is satisfied.

For $K_{li}(x,y)$ we will proceed in the similar way. We start by checking $x, y \in A$, then in this condition is satisfied we have to see whether $l_{f_A}(x)+1 = l_{f_A}(y)$ and $v_{f_A}(l_{f_A}(y), 0) = y$.

The above descriptions guarantee that $K_{s_{f_A}}$ and $K_{li_{f_A}}$ are A-recursive.

Now to define s_{f_A}, li_{f_A} we can simply write:

$$s_{f_A}(x) = \mu_{y \in A}[K_{s_{f_A}}(x,y) = 1],$$
$$li_{f_A}(x) = \mu_{y \in A}[K_{li_{f_A}}(x,y) = 1],$$

we will check the inside condition only in the case when $K_A(y) = 1$. Of course, such the definitions give us partial A-recursive functions. □

We can add that finding 'the root' x of any given element y of A i.e. the least element of $\langle A, \leq_{f_A} \rangle$ or the least limit element of $\langle A, \leq_{f_A} \rangle$ such that $y = \underbrace{s(\ldots s}_{k}(x)\ldots)$ is A-recursive operation too. We can simply define $r_{f_A}(x) = v_{f_A}(l_{f_A}(x) - 1, 0) + 1$ for $x \in A$ and $r_{f_A}(x) = 0$ otherwise. It is equally simple to define A-recursive predecessor for $x \in A$:

$$p_{f_A}(x) = \begin{cases} x & x = r_{f_A}(x), \\ v_{f_A}(l_{f_A}(x) - 1, i_{f_A}(x) - 2) & \text{otherwise.} \end{cases}$$

Now we can add the more fundamental consequence of our previous considerations.

Corollary 3.11. *For any recursively enumerable set A and its one-to--one function $f_A \in \mathrm{REC}$ we have the following restriction:*

$$\langle A, \leq_{f_A} \rangle \leq \omega^2.$$

Proof. It suffices to define the function h from $\langle A, \leq_{f_A} \rangle$ into ω^2 in this simple way:

$$h(x) = \omega \cdot (l_{f_A}(x) - 1) + i_{f_A}(x) - 1.$$

□

Let us informally observe that we have obtained results guaranteeing that the order of an infinite recursively enumerable set A given by $\langle A, \leq_{f_A} \rangle$ has to be less than ω^2. Now we will proceed with an analysis of restrictions on the orders $\langle A, \leq_{f_A} \rangle$ generated by recursiveness and non-recursiveness.

4. Ordinal numbers of recursively enumerable sets

Up to this moment we have not got any absolute mapping of recursively enumerable sets into ordinals: we have got only ordinal number for a set A relatively to a function $f_A \in \mathrm{REC}, f_A(\mathbb{N}) = A$ used to introduce well-ordering into recursively enumerable A. Let us improve this situation. As in the previous section we will consider only infinite recursively enumerable sets.

Definition 4.1. Let us consider the class $\mathcal{F}_A$ of functions $f_A \in \mathrm{REC}$ such that $f_A(\mathbb{N}) = A$ and the enumerable class $\mathcal{O}rd_A$ of order types for all well-orderings $\langle A, \leq_{f_A} \rangle$. Then we will call the least element of $\mathcal{O}rd_A$ recursive ordinal of the recursively enumerable set A and we will denote it by $\alpha(A)$.

This definition is correct because each set of ordinals has always the least element. We can start to analyse properties of $\alpha(A)$ for different sets.

Lemma 4.2. *Any infinite recursively enumerable set A is recursive if and only if $\alpha(A) = \omega$.*

Proof. ($\Leftarrow$) This is obvious: $\alpha(A) = \omega$ means there is the function $f_A \in \mathrm{REC}$ such that f_A is an increasing function, hence A is recursive.

($\Rightarrow$) If A is recursive then there is the recursive increasing function f_A such that $f_A(\mathbb{N}) = A$, hence for an infinite set A, $\alpha(A)$ has to be equal ω. □

Let us observe, that according to our constructions in the above case we have the only one level $L^0_{f_A}$.

Lemma 4.3. *If a recursively enumerable set for some $f_A \in \mathrm{REC}$ has the order type of $\langle A, \leq_{f_A} \rangle$ equal to $\omega \cdot n + k$, where $n, k \in \mathbb{N}, n \neq 0$ then $\alpha(A) = \omega$.*

Proof. Because a recursively enumerable set A satisfying the above condition can be divided into $n+1$ levels and every level corresponds to a recursive set we obtain A as the finite union of recursive sets. Of course such A has to be recursive. □

We obtain immediately the important fact.

Corollary 4.4. *Every non-recursive recursively enumerable set A has to satisfy $\alpha(A) \geq \omega^2$.*

We have proved that every non-recursive recursively enumerable set has its recursive ordinal not less than ω^2. However Corollary 3.11 gives us the inequality $\alpha(A) \leq \omega^2$ for any recursively enumerable set $A \subseteq \mathbb{N}$. Hence we obtain the final result.

Theorem 4.5. *A recursively enumerable set A is non-recursive if and only if $\alpha(A) = \omega^2$.*

Proof. If A is non-recursive recursively enumerable set then Corollaries 4.4 and 3.11 gives $\alpha(A) = \omega^2$. If A is recursively enumerable and $\alpha(A) = \omega^2$ then $\alpha(A) > \omega \cdot n + k$ for any $n, k \in \mathbb{N}$ and in that case A cannot be recursive. □

Let us recapitulate the obtained results: the above presented generalisation of monotonicity of recursive functions generating recursively enumerable sets gives us the natural ordinal ω^2. It seems possible to modify functions $\leq_{f_A}$ by regarding additional comparisons between roots of the levels; we will consider this case in the next paper presenting a different structure of ordinals for subsets of the class of recursively enumerable sets.

References

[1] P. Clote, Handbook of Computability Theory, Studies in Logic and the Foundations of Mathematics, Elsevier, 1999.

[2] T. Jech, Set Theory, Springer Monographs in Mathematics, Springer, 2006.

[3] G. Kreisel, On the interpretation of non-finitist proofs I, II, *Journal of Symbolic Logic* **16,17** (1952), 241–267, 43–58.

[4] P. Odifreddi, Classical Recursion Theory, Studies in Logic and the Foundations of Mathematics, North Holland, 1989.

[5] J. Roitman, Introduction to Modern Set Theory, Virginia Commonwealth University, 2011.

[6] R. I. Soare, Recursively Enumerable Sets and Degrees, Perspectives in Mathematical Logic, Springer, 1987.

[7] S. S. Wainer, A classification of the ordinal recursive functions, *Archiv fur Mathematische Logik und Grundlagenforschung* **13**:3–4 (1970), 136–153.

[8] R. Weber, Computability Theory, Student Mathematical Library, American Mathematical Society, 2012.

Institute of Mathematics
University of Maria Curie-Skłodowska
pl. M. Curie-Skłodowskiej 1
20-709 Lublin, Poland
jerzy.mycka@umcs.lublin.pl