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BASIC PRINCIPLES AND THEOREMS  
OF DIMENSIONAL ANALYSIS AND THE THEORY  
OF MODEL SIMILARITY OF PHYSICAL PHENOMENA

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Abstract

The paper concerns dimensional analysis and the theory of model similarity of physical phenomena. At the beginning of the considerations, basic notions, definitions, relationships and fundamental principles of the issues to be analysed has been presented. Next, Buckingham's  $\Pi$ -theorem of dimensional analysis and theory of similarity as well as authorial generalized theorems  $\Pi$  of this field of interest have been derived. A separate chapter has been devoted to the problem of the nature of physical phenomena occurring in the mechanics of continuous or discrete material mediums. At the end of the paper, an example of the determination of similarity numbers in the case of a system with one degree of freedom at mechanical and kinematic excitation have been given.

*Keywords: dimensional analysis, model similarity, physical phenomena, principles and theorems*

Streszczenie

Praca dotyczy analizy wymiarowej i teorii podobieństwa modelowego zjawisk fizycznych. Na wstępie przedstawiono podstawowe pojęcia, definicje, związki i fundamentalne zasady dotyczące rozpatrywanych problemów. Następnie wyprowadzono twierdzenie  $\Pi$  Buckingham'a dotyczące analizy wymiarowej i teorii podobieństwa oraz autorskie uogólnienie twierdzenia  $\Pi$ . Oddzielny rozdział został poświęcony problemowi zjawisk fizycznych występujących w mechanice ośrodków ciągłych i dyskretnych. W części końcowej artykułu przedstawiono przykład określenia liczb podobieństwa w przypadku układu o jednym stopniu swobody przy wzbudzeniu mechanicznym i kinematycznym.

*Słowa kluczowe: analiza wymiarowa, podobieństwo modelowe, zjawiska fizyczne, zasady i twierdzenia*

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## 1. Introduction

Dimensional analysis offers a method for reducing complex physical problems to the simplest form prior to obtaining a quantitative answer.

The method is of great generality and mathematical simplicity. At the heart of dimensional analysis is the concept of similarity. In physical terms, similarity refers to some equivalence between two things, processes or phenomena that are actually different. Different with respect to nature or scale, processes or phenomena.

Mathematically, similarity refers to a transformation of variables that leads to a reduction in the number of independent variables that specify the problem. A problem that at first looks formidable may sometimes be solved with little effort after dimensional analysis.

In problems so well understood that one can write down in mathematical form all the governing laws and boundary conditions, and only the solution is lacking, similarity can also be inferred by normalizing all the equations and boundary conditions in terms of quantities that specify the problem and identifying the dimensionless groups that appear in the resulting dimensionless equations. This is an inspectional form of similarity analysis.

Dimensional analysis is, however, the only option in problems where the equations and boundary conditions are not completely articulated, and always useful because it is simple to apply and quick to give insight.

Some of the basic ideas of similarity and dimensional analysis had already presented in Fourier's work in the first quarter of the nineteenth century, but the subject received more methodical attention only toward the close of that century, notably in the works of Lord Rayleigh, Reynolds, Maxwell, and Froude in England, and Carvallo, Vaschy and a number of other scientists and engineers in France [22, 24]. By the 1920's Buckingham's now well-known  $\Pi$ -theorem had appeared [5] and Bridgman had published the monograph which still remains the classic in the field [4]. Since then, the literature has grown enormously. Applications now include different branches of knowledge and science. The procedure is the same in all applications, a great variety of which may be found in the references and in the scientific literature at large (see for example following books, handbooks and monographs: [2–3, 6, 8–21, 23, 25–34, 36–39].

## 2. Physical quantities and relationships

### 2.1. Physical properties

An object or event or phenomenon is described in terms of basic properties like length, mass, colour, shape, speed, and time. None of these properties can be defined in absolute terms. We can do no more than compare one thing with another.

A physical property first arises as a concept based on experience and is formalized by defining a comparison operation for determining whether two samples of it are equal ( $A=B$ ) or unequal ( $A\neq B$ ).

This operation, which is an entirely physical procedure, defines the property. Properties of the same kind are compared by means of the same comparison operation. Properties of different kinds cannot be compared. Asking whether a particular mass is physically equal to a particular length is meaningless: no procedure exists for making the comparison.

Properties like shape and colour are useful for describing things, but cannot play a role in any quantitative analysis.

## 2.2. Physical quantities and base quantities

Science begins with observation and description of things and phenomena. Its ultimate goal is to infer from those observations laws that express the phenomena of the physical world in the simplest and most general terms. The language of mathematics is ideally suited for expressing those laws. The allowed types of properties are called “physical quantities”.

Physical quantities are of two types: base quantities and derived quantities. The base quantities form a complete set of derived quantities that may be introduced as necessary. The base and derived quantities together provide a rational basis for describing and analysing the physical world in quantitative terms.

A base quantity is defined by specifying two physical operations:

1. A comparison operation for determining whether two samples  $A$  and  $B$  of the property are equal ( $A=B$ ) or unequal ( $A\neq B$ );
2. An addition operation that defines what is meant by the sum  $C=A+B$  of two samples of the property.

Base quantities with the same comparison and addition operations are of the same kind. The addition operation  $A+B$  defines a physical quantity  $C$  of the same kind as the quantities being added. All physical quantities are properties of physical things, events, processes, or phenomena.

The comparison and addition operations are physical, but they are required to have certain properties that mimic those of the corresponding mathematical operations for pure numbers [35]:

1. The comparison operation must obey the identity law (if  $A=B$  and  $B=C$ , then  $A=C$ );
2. The addition operation must be commutative ( $A+B=B+A$ ), associative [ $A+(B+C)=(A+B)+C$ ], and unique (if  $A+B=C$ , there exists no finite  $D$  such that  $A+B+D=C$ ).

The two operations together define, in entirely physical terms:

1. The concept of larger and smaller for like quantities (if there exists a finite  $B$  such that  $A+B=C$ , then  $C>A$ );
2. Subtraction of like quantities (if  $A+B=C$ , then  $A\equiv C-B$ );
3. Multiplication of a physical quantity by a pure number (if  $B=A+A+A$ , then  $B\equiv 3A$ );
4. Division of a physical quantity by a pure number (if  $A=B+B+B$ , then  $B\equiv A/3$ ).

A base quantity is thus a property for which the following mathematical operations are defined in physical terms: comparison, addition, subtraction, multiplication by a pure number, and division by a pure number. Each of these operations is performed on physical properties of the same kind and yields a physical property of that kind, and each physical operation obeys the same rules as the corresponding mathematical operation for pure numbers.

It is important to note that mathematical operations other than the ones listed above are not defined in physical terms. Products, ratios, powers, and exponential and other functions such as trigonometric functions and logarithms are defined for numbers, but have no physical correspondence in operations involving actual physical quantities.

### 2.3. Measure unit, dimension and numerical value

The two operations that define a base quantity make it possible to express any such quantity as a multiple of a standard sample of its own kind, i.e. a unit of measure or simply – a unit. The standard sample – the unit – may be chosen arbitrarily.

The measuring process consists of physically adding replicas of the unit and fractions thereof until the sum equals the quantity being measured. A count of the number of whole and fractional units required yields the numerical value of the quantity being measured. If  $a$  is the unit chosen for quantities of type  $A$ , the process of measurement yields a numerical value  $\check{A}$  (a number) such that:

$$A = \check{A}a \quad (2.1)$$

In further considerations the unit of measure  $a$  will be called the dimension of the quantity  $A$  and will be described as:

$$a = [A]; \quad A = \check{A}[A] \quad (2.2)$$

The numerical value of a base quantity depends on the choice of unit. A physical quantity exists independently of the choice of unit. A quantity  $A$  can be measured in terms of a unit  $a$  or in terms of another unit  $a'$ , but the quantity itself remains physically the same, that is:

$$A = \check{A}a = \check{A}[A] = \check{A}'a' = \check{A}'[A'] \quad (2.3)$$

If the unit  $[A']$  is times larger than  $[A]$ :

$$[A'] = \check{N}[A], \quad (2.4)$$

it follows from equation (2.3) that:

$$\check{A}' = \check{N}^{-1}\check{A}. \quad (2.5)$$

If the size of a base quantity's unit is changed by a factor  $\check{N}$ , the quantity's numerical value changes by a factor  $\check{N}^{-1}$ .

The ratio of the numerical values of any two base quantities of the same kind is independent of base unit size.

Note also that when base quantities of the same kind are added physically ( $A+B=C$ ), the numerical values satisfy an equation of the same form as the physical quantity equation

$(A+B=C)$ , regardless of the size of the chosen unit. In other words, the numerical value equation mimics the physical equation, and its form is independent of the unit's size.

## 2.4. Fundamental measure unit base

Set of  $k$  base quantities  $\{A\}=\{A_1, \dots, A_k\}$  which are one dimensionally independent, i.e. none of its members has a dimension that can be expressed in terms of the dimensions of the remaining members, and which are sufficient and complete to describe the dimensions all other quantities involving in describing a respective object, event or physical phenomenon, is called fundamental measure unit base or shortly fundamental unit base. For example in mechanics such a base contains three base quantities:  $L$  – length,  $M$  – mass,  $T$  – time. In thermomechanics problems another base quantity appears, i.e.  $\tau$  – temperature.

## 2.5. Derived quantities, their dimensions and dimensionless quantities

Describing physical things, events or phenomena quantitatively, we refer to numerical values of base quantities and also introduce numbers derived by inserting these values into certain mathematical formulas, expressions, relationships, etc.

Derived quantities by definition are those quantities which satisfy the following rules:

1. An arbitrary derived quantity  $Q$  can be presented in a form:

$$Q = \tilde{Q}[Q] \quad (2.6)$$

where  $\tilde{Q}$  is the numerical value of the  $Q$  and  $[Q]$  is the dimension of the  $Q$ .

2. Numerical values of derived quantities are defined by respective mathematical formulas, expressions, relationships, etc. containing mathematical operators such as: algebraic operators, functional operators, derivative operators, integral operators, operators like:  $\lim$ ,  $\sum$ , etc., in which numerical values of base quantities appear.
3. Numerical values of derived quantities can be presented in a power-law form:

$$Q = \tilde{q} A_1^{\alpha_{Q1}} \dots A_k^{\alpha_{Qk}} \quad (2.7)$$

where  $\tilde{q}$  is a dimensionless quantity (number) and powers:  $\alpha_{Q1}, \dots, \alpha_{Qk}$  are real numbers, whose values distinguish one type of derived quantity from another. All monomial derived quantities have this power-law form; no other form represents a physical quantity.

4. The dimension  $[Q]$  of the derived quantity  $Q$  mimics the mathematical formula for the numerical value of the quantity  $Q$  omitting the number  $\tilde{q}$ , i.e.

$$[Q] = [A_1]^{\alpha_{Q1}} \dots [A_k]^{\alpha_{Qk}} \quad (2.8)$$

5. The derived quantity  $Q$  is defined in terms of the numerical value  $Q$  which depends on the choice of base units.

Whether applied to a base or derived quantity, the dimension is simply a formulaic indication of how the quantity's numerical value transforms when the sizes of the base units are changed. A derived quantity's dimension follows from its defining equation. We simply substitute for each base quantity the symbol for its dimension, omit the numerical coefficient  $\bar{q}$  and obtain the equation by algebra. Thus, a quantity's dimension depends on the choice of the system of units.

Base quantities have a transparently physical origin, which gives rise to the fact that the ratio of any two samples of a base quantity remains constant when the base unit size is changed. Bridgman [4] postulated that this is in fact a defining attribute of all physical quantities, both base and derived quantities. This is Bridgman's principle of absolute significance of relative magnitude: A number  $Q$ , obtained by inserting the numerical values of base quantities into a formula, is a physical quantity if the ratio of any two samples of it remains constant when base unit sizes are changed.

Bridgman went on to show [4] (see also the proof by Barenblatt [1] and others [32, 38]) that a monomial formula satisfies the principle of absolute significance of relative magnitude.

Some important points about derived quantities can be listed as follows [35]:

1. The dimension of any derived physical quantity is a product of powers of the base quantity dimensions.
2. Sums of derived quantities with the same dimension are derived quantities of the same dimension. Products and ratios of derived quantities are also derived quantities with dimensions which are usually different from the original quantities.
3. All derived quantities with the same dimension change their values by the same factor when the sizes of the base units are changed.
4. A derived quantity is dimensionless if its numerical value remains invariant when the base units are changed. An example is  $Vt/L$ , where  $V = dx/dt$  is a velocity,  $t$  is a time and  $L$  is a length. The dimension of a dimensionless quantity is unity, the factor by which the quantity's numerical value changes when base unit sizes are changed.
5. Special functions (logarithmic, exponential, trigonometric, etc.) of dimensional derived quantities are in general not derived quantities because their values do not in general transform like derived quantities when base unit size changes. Only when the arguments of these functions are dimensionless are the values of the functions remain invariant when units changed. Special functions with dimensionless arguments are therefore derived quantities with dimension unity.

## 2.6. System of units

A system of units is defined by:

1. A complete set of base quantities with their defining comparison and addition operations;
2. The base units;
3. All relevant derived quantities, expressed in terms of their defining equations.

The set of derived quantities is open-ended; new ones may be introduced in some new problems and analyses.

Systems of units are said to be of the same type if they differ only in the magnitudes of the base units. In the SI system (Système International) there are six base quantities (Tab. 1): length, time, mass, temperature, current, number of elementary particles, and luminous intensity. The units of length, time and mass are the metre (m), the second (s) and the kilogram (kg), respectively. Force is considered a derived quantity by writing Newton's law as  $F = ma$ .

Also sometimes included among the base quantities are two dimensionless quantities, plane angle and solid angle, which are measured in radians and steradians, respectively. We consider them derived quantities because, though dimensionless, they are defined in terms of operations involving length, much like area is defined in terms of length operations. The SI system of units – derived quantities (incomplete set) is given in Tab. 2.

Table 1

**The SI system of units – base quantities (complete set)**

Quantity	SI name	SI Symbol
length, $L$	metre	M
time, $t$	second	S
mass, $M$	kilogram	kg
temperature, $T$	Kelvin	K
current, $I$	ampere	A
number of elementary particles	Mole	mol
luminous intensity	candela	cd

Table 2

**The SI system of units – derived quantities (incomplete set)**

Quantity	Defining equation/law	Dimension	Dimensional Symbol	Name
area	$A = \int dx dy$	$L^2$	$m^2$	---
volume	$V = \int dx dy dz$	$L^3$	$m^3$	---
frequency	$f = 1/\tau$	$t^{-1}$	$s^{-1}$	hertz (Hz)
velocity	$v = dx/dt$	$Lt^{-1}$	$ms^{-1}$	---
acceleration	$a = d^2x/dt^2$	$Lt^{-2}$	$ms^{-2}$	---
density	$\rho = M/V$	$ML^{-3}$	$kgm^{-3}$	---
force	$F = Ma$	$MLt^{-2}$	$kgms^{-2}$	newton (N)
stress/pressure	$p = F/A$	$ML^{-1}t^{-2}$	$Nm^{-2}=kgm^{-1}s^{-2}$	pascal (Pa)
work/energy	$W = \int F dx$	$ML^2t^{-2}$	$Nm=kgm^2s^{-2}$	joule (J)
torque	$T = Fl$	$ML^2t^{-2}$	$Nm=kgm^2s^{-2}$	---
power	$dW/dt$	$ML^2t^{-3}$	$Js^{-1}=kgm^2s^{-3}$	watt (W)
charge	$Q = \int I dt$	$It$	$As$	coulomb (C)

It is important to point out that the dimension of a derived quantity depends on the choice of system of units, which is under the control of the observer and has nothing to do with the quantity's intrinsic nature. Indeed, quantities with quite different physical meaning, like work and torque, can have the same dimension.

### 2.7. Physical quantities dimensionally dependent and independent. Derived dimensional base

All physical quantities have dimensions which can be expressed as products of powers of the set of base dimensions. Alternatively, it is possible to express the dimension of one quantity as a product of powers of the dimensions of other quantities which are not necessarily base quantities.

Let us consider a set of physically independent dimensional variables  $\{Q_1, \dots, Q_i, \dots, Q_n\}$ ;  $i = 1, 2, \dots, n$ . The variables  $Q_i$  are independent if the numerical value of each member can be adjusted arbitrarily without affecting the numerical value of any other member. This set is put in order in such a way that it is possible to pick out from the physically independent variables  $Q_1, \dots, Q_n$  a dimensionally independent and complete subset  $\{Q_1, \dots, Q_k\}$ , i.e. one which satisfies the following features:

1. The dimensions of this subset contain dimensions of the fundamental units base  $\{A_1, \dots, A_k\}$ .
2. The size ( $k$ ) of the subset  $\{Q_1, \dots, Q_k\}$  is the same as the size of the fundamental base  $\{A_1, \dots, A_k\}$ .
3. None of its members has a dimension that can be expressed as product of powers of dimensions of the remaining members.
4. Dimensions of all the remaining members i.e.  $Q_{k+1}, \dots, Q_n$  can be expressed as product of powers of dimensions of the subset members i.e.  $Q_1, \dots, Q_k$ . Such a subset  $\{Q_1, \dots, Q_k\}$  will be called the derived dimensional base of the set  $\{Q_1, \dots, Q_n\}$ .

It can be proven that for the set  $\{Q_1, \dots, Q_n\}$  such a derived dimensional base exists and in general there is more than one derived dimensional base for the initial set  $\{Q_1, \dots, Q_n\}$ .

### 2.8. Physical relationships. Dimensional homogeneity

Science is concerned only with expressing physical relationships between quantities characterizing different phenomena. In quantitative analysis of physical phenomena (objects, events, processes) one seeks mathematical relationships (expressions, functions, equations, inequalities) between the numerical values of the physical quantities that describe the phenomenon.

Nature is indifferent to the arbitrary choices of base units. So, we are interested only in numerical relationships that remain true independent of base unit size.

This puts certain constraints on the allowable form of physical relationships. In other words, a physical relationships must be dimensionally homogeneous. Dimensional homogeneity



imposes the following constraints on any mathematical representation of a relationship:

1. Both sides of the relationship must have the same dimension;
2. Wherever a sum of quantities appears in relationship, all the terms in the sum must have the same dimension;
3. All arguments of any exponential, logarithmic, trigonometric or other special functions that appear in functions must be dimensionless.

For example, if physical equation is represented by:

$$A = Be^{-c} - \frac{(D_1 + D_2)}{E} + F \sin(\Omega t + \varphi) + G \log_{10} \left( \frac{H}{J} \right), \quad (2.9)$$

$C$  must be dimensionless,  $D_1$  and  $D_2$  must have the same dimension,  $A$ ,  $B$ ,  $D/E$ ,  $F$  and  $G$  must have the same dimension,  $(\Omega t + \varphi)$  and  $H/J$  must be dimensionless.

An important consequence of dimensional homogeneity is that the forms of a physical relationships are independent of the size of the base units.

Every correct physical function, equation, inequality – that is, every relationship that expresses a physically significant dependence between numerical values of physical quantities – must be dimensionally homogeneous.

## 2.9. Recapitulation

The most important inferences, conclusions and statements of the considerations presented in p. 2 can be formulated as follows [35]:

1. A base quantity is a property that is defined in physical terms by two operations: a comparison operation, and an addition operation.
2. Base quantities are properties for which the following concepts are defined in terms of physical operations: equality, addition, subtraction, multiplication by a pure number, and division by a pure number. Not defined in terms of physical operations are: product, ratio, power, and logarithmic, exponential, trigonometric and other special functions of physical quantities.
3. A base quantity can be measured in terms of an arbitrarily chosen unit of its own kind and a numerical value.
4. A derived quantity is defined in terms of numerical value (which depends on base unit size) and dimension. Both are defined by power-law formula (2.7) and (2.8).
5. The same quantity (e.g. force) may have different dimensions in different systems of units, and quantities that are clearly physically different (e.g. work and torque) may have the same dimension.
6. Relationships between physical quantities may be represented by mathematical relationships between their numerical values. A mathematical expression, function, equation, inequality etc. that correctly describes a physical relationship between quantities is dimensionally homogeneous.
7. A system of units is defined by (a) the base quantities, (b) their units, and (c) the derived quantities. Both the type and the number of base quantities are open to choice.

### 3. Fundamental principles of dimensional analysis and model similarity of physical phenomena

These fundamental principles are listed below:

1. Real or postulated physical laws which are used in descriptions of physical phenomena should be objective, i.e. independent of the system of units adopted. It must be possible to transform a dimensional set of physical dependencies and then mathematical relationships (functions, formulas, expressions, equations, inequalities, etc.) which describe any physical phenomenon (more precisely – physical model and then mathematical model of that phenomenon) into a dimensionless form, in which the form does not depend on the system of units. Among all dimensional quantities characterizing some physical phenomenon it must be possible to create dimensionless quantities which appear in such relationships. This also concerns all mathematical operators which appear in these relationships. Moreover, dimensionless relationships describing some physical phenomenon must satisfy the principle of dimensional homogeneity too – as was pointed out previously.
2. Two physical phenomena are similar if:
  - The structure of the relationships and mathematical operators used that describe these phenomena are similar;
  - Dimensionless form of relationships describing these phenomena is similar.
3. Two types of similarity phenomena are distinguished:
  - Similarity of analogy type similarity, where two phenomena have different physical natures (e.g. mechanical and electrical oscillations one degree of freedom systems presented in Fig. 1.)
  - Similarity of model type similarity, where two physical phenomena have a similar physical nature but they are different with respect to the scale of this phenomenon (e.g. phenomenon in the natural scale and a phenomenon model in a smaller scale, but scales of particular physical quantities characterizing this phenomena can be different). In further considerations the second case will be considered, i.e. model similarity.

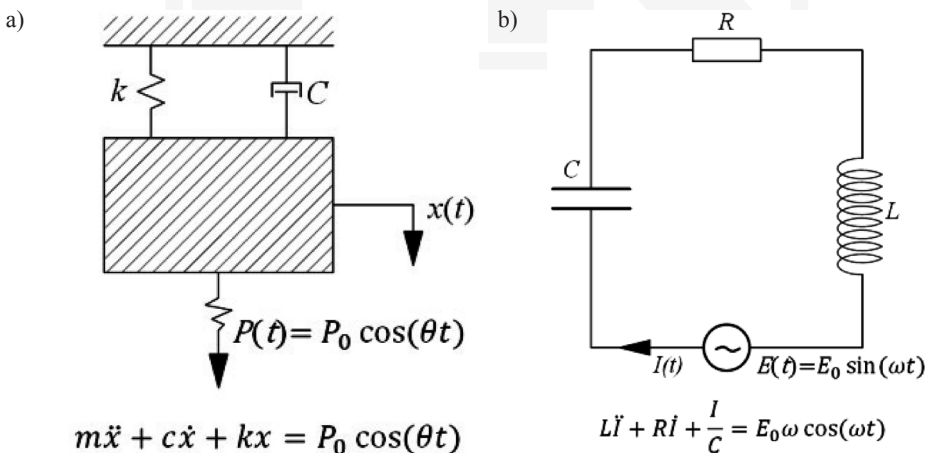


Fig. 1. One degree of freedom mechanical system (a) and its electrical analogy (b)

4. In dimensionless form relationships describing a given physical phenomenon at the natural scale ( $N$ ) and the model scale (smaller) ( $M$ ) appear dimensionless numbers dependent on the particular physical quantities characterizing that phenomenon and its scale. In general, the values of these numbers at the natural scale and the model scale are different. To make the phenomena similar, these numbers at the natural and model scales should be equal to each other. These are the model similarity criteria of both phenomena. The numbers are called similarity numbers. Their quotient – respectively at the natural and model scales should be equal to one. Hence, similarity criteria (or relationships) can be determined for the scales of different physical quantities characterizing a given physical phenomenon.
5. There is usually a cause and effect nature of physical phenomena. In this case, among all the dimensional and dimensionless physical quantities, characterizing a given cause and effect physical phenomenon, it is always possible to separate physical quantities connected with input – IN (action on system), system – O (material object), and output – OU (response of system, reaction of system). Then, physical phenomena can be described and analysed using terminology, notions, concepts and block diagrams of systems analysis treating physical phenomena as input/output physical systems. In the case of compound systems, there are in general several inputs, objects and outputs.

The essence of model investigations is to determine from the measurements dimensionless quantities of output (OU) at given quantities of input (IN) and system (O) and with criteria satisfying total or partial similarity. The numerical values of dimensionless output quantities obtained can then be transferred to the real system (natural scale) as the resulting similarity criteria numbers.

#### 4. Buckingham's $\Pi$ -theorem of dimensional analysis and theory of similarity

##### 4.1. The steps of dimensional analysis and Buckingham's $\Pi$ -theorem

The form of any physically significant function, equation, or inequality must be such that the relationship between the actual physical quantities remains valid independent of the magnitudes of the base units. Dimensional analysis derives the logical consequences of this premise. Buckingham's  $\Pi$ -theorem, which follows from dimensional analysis, can be performed using the four steps of dimensional analysis presented below [35].

Suppose we are interested in some particular physical quantity  $Q_0$  that is a "dependent variable" in a well-defined physical process or event. By this we mean that, once all the quantities that define the particular process or event are specified, the value of  $Q_0$  follows uniquely.

##### **Step 1: The independent variables**

The first and most important step in dimensional analysis is to identify a complete set of independent quantities  $Q_1 \dots Q_n$  that determine the value of  $Q_0$ :

$$Q_0 = f(Q_1, Q_2, \dots, Q_n) \quad (4.1)$$

Starting with the correct set  $Q_1 \dots Q_n$  is as important in dimensional analysis as it is in mathematical physics to start with the correct fundamental equations and boundary conditions.

The relationship expressed symbolically in equation (4.1) is the result of the physical laws that govern the phenomenon of interest.

### Step 2: Dimensional considerations

Next we list the dimensions of the dependent variable  $Q_0$  and the independent variables  $Q_1 \dots Q_n$ . We must specify at least the type of the system of units before we do this. For example, in a purely mechanical problem, all quantities have dimensions of the form:

$$[Q_i] = [L]^{l_i} [M]^{m_i} [T]^{t_i}; \quad i = 0, 1, 2, \dots, n, \quad (4.2)$$

where the exponents  $l_i$ ,  $m_i$  and  $t_i$  are dimensionless numbers that follow from each quantity's definition.

We now pick from the complete set of physically independent variables  $Q_1 \dots Q_n$  a complete, dimensionally independent subset  $Q_1 \dots Q_k$  ( $k \leq n$ ), and express the dimension of each of the remaining independent variables  $Q_{k+1} \dots Q_n$  and the dependent variable  $Q_0$  as a product of the powers of  $Q_1 \dots Q_k$ .

Since equation (4.1) is dimensionally homogeneous, the dimension of the dependent variable  $Q_0$  is also expressible in terms of the dimensions of  $Q_1 \dots Q_k$ .

The dimensionally independent subset  $Q_1 \dots Q_k$  may be selected in different ways, but the number  $k$  of dimensionally independent quantities in the full set  $Q_1 \dots Q_n$  is unique to the set and cannot exceed the number of base dimensions which appear in the dimensions of the quantities in that set. For example, if the dimensions of  $Q_1 \dots Q_n$  involve only length, mass, and time, then  $k \leq 3$ .

Having chosen a complete, dimensionally independent subset  $Q_1 \dots Q_k$ , we express the dimensions of  $Q_0$  and the remaining quantities  $Q_{k+1} \dots Q_n$  in terms of the dimensions of  $Q_1 \dots Q_k$ . These will have the form:

$$[Q_j] = [Q_1]^{\alpha_{j1}} [Q_2]^{\alpha_{j2}} \dots [Q_k]^{\alpha_{jk}} \quad (4.3)$$

if  $j > k$  or  $j = 0$ . The exponents are dimensionless real numbers and  $l = 1, 2, \dots, k$ .

Let us take  $Q_1$ ,  $Q_2$ , and  $Q_3$  as the complete dimensionally independent subset. Equating the dimension given by equation (4.2) with that of equation (4.3), we obtain three equations:

$$l_j = \sum_{l=1}^3 \alpha_{jl} l_l \quad m_j = \sum_{l=1}^3 \alpha_{jl} m_l \quad t_j = \sum_{l=1}^3 \alpha_{jl} t_l \quad (4.4)$$

which can be solved for the three unknowns  $\alpha_{j1}$ ,  $\alpha_{j2}$ , and  $\alpha_{j3}$ .

### Step 3: Dimensionless variables

We now define dimensionless forms of the  $n-k$  remaining independent variables by dividing each one with the product of the powers of  $Q_1 \dots Q_k$  which has the same dimension:

$$\tilde{\Pi}_j = \frac{Q_j}{Q_1^{\alpha_{j1}} + Q_2^{\alpha_{j2}} \dots Q_k^{\alpha_{jk}}} \quad (4.5)$$

where  $j = k+1, \dots, n$ , and a dimensionless form of the dependent variable  $Q_0$ :

$$\check{\Pi}_0 = \frac{Q_0}{Q_1^{\alpha_{01}} + Q_2^{\alpha_{02}} \dots Q_k^{\alpha_{0k}}} \quad (4.6)$$

#### Step 4: The end game and Buckingham's $\Pi$ -theorem

An alternative form of equation (4.1) is:

$$\check{\Pi}_0 = f^*(Q_1, Q_2, \dots, Q_k; \check{\Pi}_{k+1}, \check{\Pi}_{k+2}, \dots, \check{\Pi}_n) \quad (4.7)$$

in which all quantities are dimensionless except  $Q_1 \dots Q_k$ . The values of the dimensionless quantities are independent of the sizes of the base units. The values of  $Q_1 \dots Q_k$ , on the other hand, do depend on base unit size. They cannot be put in dimensionless form since they are (by definition) dimensionally independent of each other. From the principle that any physically meaningful relationships (i.e. the function in the case analysed) must be dimensionally homogeneous, that is, valid independent of the sizes of the base units, it follows that  $Q_1 \dots Q_k$  must in fact be absent from equation (4.7), that is:

$$\check{\Pi}_0 = \check{f}(1, 1, \dots, 1; \check{\Pi}_{k+1}, \check{\Pi}_{k+2}, \dots, \check{\Pi}_n) = \check{f}(\check{\Pi}_{k+1}, \check{\Pi}_{k+2}, \dots, \check{\Pi}_n) \quad (4.8)$$

This equation is the final result of the dimensional analysis, and is the base of Buckingham's  $\Pi$ -theorem:

When a complete relationship between dimensional physical quantities is expressed in dimensionless form, the number of independent quantities that appear in it is reduced from the original  $n$  to  $n-k$ , where  $k$  is the number of the dimensional base size, which reduces the number of independent quantities in the problem by  $k$  and simplifies the problem enormously.

The  $\Pi$ -theorem itself merely tells us the number of dimensionless quantities that affect the value of a particular dimensionless dependent variable. It does not tell us the forms of the dimensionless variables. That form has to be discovered by experimentation or by solving the problem theoretically knowing the mathematical model of the problem.

Dimensionless numbers  $\check{\Pi}_j$ ;  $j=k+1, k+2, \dots, n$  are independent of the system of units and constitute specified similarity criteria (similarity criteria numbers) of a given physical phenomenon with other similar to its physical phenomenon. In the case of two phenomena  $I$  and  $II$  of different physical nature but similar in terms of analogy type similarity:  $\check{\Pi}_j^I = \check{\Pi}_j^{II}$ . In the case of two physical phenomena of the same physical nature but of two scales: real ( $N$ ) and model ( $M$ ), similar in terms of model similarity:  $\check{\Pi}_j^N = \check{\Pi}_j^M$ .

In the case when an initial functional relationship is performed in a form of implicit function, i.e.:

$$F(Q_0, Q_1, \dots, Q_n) = 0 \quad (4.9)$$

where respective dimension quantities  $Q_0, Q_1, \dots, Q_n$  are quantities dependent or independent of each other, the successive steps of dimensional analysis leading to the theorem  $\Pi$  are similar as before and final result is as follows:

$$\check{F}(\check{\Pi}_0, \check{\Pi}_1, \dots, \check{\Pi}_{n-k}) = 0 \quad (4.10)$$

4.2. The second method of reaching Buckingham's Π-theorem

There are  $n$  dimensional physical quantities:  $Q_1, Q_2, \dots, Q_n$ ; some of these may be dependent on others. Furthermore, there is a functional relationship between these quantities:

$$F(Q_1, Q_2, \dots, Q_n) = 0 \tag{4.11}$$

Moreover, one may consider the basic dimensional base of the physical phenomenon described by relationship (4.11) i.e.  $\{A_1, A_2, \dots, A_k\}$ . Dimension of any dimensional quantity  $Q_i$ ;  $i = 1, 2, \dots, n$  can be performed as a product of the powers of base quantities  $A_l$ ,  $l = 1, 2, \dots, k$  as:

$$\begin{aligned} [Q_1] &= [A_1]^{\alpha_{11}} [A_2]^{\alpha_{21}} \dots [A_k]^{\alpha_{k1}} \\ [Q_2] &= [A_1]^{\alpha_{12}} [A_2]^{\alpha_{22}} \dots [A_k]^{\alpha_{k2}} \\ &\dots \\ [Q_n] &= [A_1]^{\alpha_{1n}} [A_2]^{\alpha_{2n}} \dots [A_k]^{\alpha_{kn}} \end{aligned} \tag{4.12}$$

The basic issue can be formulated as follows: are there any dimensionless expressions in the form:

$$\tilde{\Pi} = Q_1^{q_1} Q_2^{q_2} \dots Q_k^{q_k}, \tag{4.13}$$

where  $q_1, q_2, \dots, q_k$  – real numbers, and how many such numbers  $\tilde{\Pi}$  are there in the particular case? Since:

$$[\tilde{\Pi}] = [A_1]^0 [A_2]^0 \dots [A_k]^0 = \left( [A_1]^{\alpha_{11}} [A_2]^{\alpha_{21}} \dots [A_k]^{\alpha_{k1}} \right)^{q_1} \dots \left( [A_1]^{\alpha_{1n}} [A_2]^{\alpha_{2n}} \dots [A_k]^{\alpha_{kn}} \right)^{q_n} \tag{4.14}$$

hence, by comparing the exponents of base quantities, it can be obtained:

$$\begin{aligned} \alpha_{11}q_1 + \alpha_{12}q_2 + \dots + \alpha_{1n}q_n &= 0 \\ \alpha_{21}q_1 + \alpha_{22}q_2 + \dots + \alpha_{2n}q_n &= 0 \\ &\dots \\ \alpha_{k1}q_1 + \alpha_{k2}q_2 + \dots + \alpha_{kn}q_n &= 0 \end{aligned} \tag{4.15}$$

This is a homogeneous system of linear equations consisting of  $k$  equations of  $n$  unknowns.

The question of the system's solvability comes down thereby to the question of rank of the following dimensional matrix:

	$Q_1$	$Q_2$	$\dots$	$Q_n$
$A_1$	$\alpha_{11}$	$\alpha_{12}$	$\dots$	$\alpha_{1n}$
$A_2$	$\alpha_{21}$	$\alpha_{22}$	$\dots$	$\alpha_{2n}$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$A_k$	$\alpha_{k1}$	$\alpha_{k2}$	$\dots$	$\alpha_{kn}$

As we know, the rank of a matrix is expressed as  $r$  if it consists of one determinant different from zero of the rank  $r$  at least, while the other determinants of higher order are zeros. So, it is correct to use the theorem from linear algebra to the relationships (4.15): homogeneous, linear set of equations with  $n$  unknowns, whose matrix has coefficient of rank  $r$ , have exactly  $n-r$  independent solutions. With respect to our basic problem of dimensional analysis it looks as follows (Buckingham's  $\Pi$ -theorem): There are  $n$  quantities  $Q_1, Q_2, \dots, Q_n$  and a dependence (4.11) between them. Thus, there exist exactly  $n-r$  dimensionless quantities  $\tilde{\Pi}$ , wherein the rank of dimensional matrix is  $r \leq m \leq n$ . Dependence:  $\tilde{F}(\tilde{\Pi}_1, \tilde{\Pi}_2, \dots, \tilde{\Pi}_{n-r}) = 0$  is the solution of this issue.

### 4.3. The third (original) method of reaching Buckingham's $\Pi$ -theorem

For dependent and independent dimensional quantities  $Q_1, Q_2, \dots, Q_n$  appearing in the functional relationship  $F(Q_1, Q_2, \dots, Q_n) = 0$  it dimensionless quantities  $Q_{ref,1}, Q_{ref,2}, \dots, Q_{ref,n}$  can be created, assuming for each of these quantities  $Q_i, i = 1, 2, \dots, n$  some reference quantities  $Q_{ref,i}$  in the following way:

$$\tilde{Q}_{ref,i} = \frac{Q_i}{Q_{ref,i}} \quad (4.16)$$

And then the starting functional relationship can be performed in such a form:

$$F(\tilde{Q}_{ref,1} Q_{ref,1}, \tilde{Q}_{ref,2} Q_{ref,2}, \dots, \tilde{Q}_{ref,n} Q_{ref,n}) = 0 \quad (4.17)$$

where  $\tilde{Q}_{ref,i} Q_{ref,i}$  quantities are ordered in such a way that among the first few  $k$  quantities  $Q_{ref,l}, l=1, 2, \dots, k$  it is possible to create the complete dimensional base of the problem. Thus, the dimensions of the other quantities  $Q_{ref,j}, j=k+1, k+2, \dots, n$  (there is number  $n-k$  of them) can be performed in such a form:

$$[Q_{ref,j}] = [Q_{ref,1}]^{\alpha_{ref,1j}} [Q_{ref,2}]^{\alpha_{ref,2j}} \dots [Q_{ref,k}]^{\alpha_{ref,kj}} \quad (4.18)$$

Hence, the following dimensionless quantities (numbers)  $\tilde{\Pi}_{Q_{ref,j}}^*$  and  $\tilde{\Pi}_{Q_{ref,j}}$  can be defined:

$$\tilde{\Pi}_{Q_{ref,j}}^* = \frac{Q_{ref,j}}{Q_{ref,1}^{\alpha_{ref,1j}} Q_{ref,2}^{\alpha_{ref,2j}} \dots Q_{ref,k}^{\alpha_{ref,kj}}} \quad (4.19)$$

$$\tilde{\Pi}_{Q_{ref,j}} = \tilde{Q}_{ref,j} \tilde{\Pi}_{Q_{ref,j}}^* \quad (4.20)$$

Then function (4.17) can also be written in another form, namely:

$$F^* \left( \{ \dots, \tilde{Q}_{ref,l}, \dots \}; \{ \dots, Q_{ref,l}, \dots \}; \{ \dots, \tilde{\Pi}_{Q_{ref,j}}, \dots \} \right) = 0$$

$$l = 1, 2, \dots, k; j = k + 1, \dots, n \quad (4.21)$$

Since the physical laws and all the other relationships connected with them should be objective, they must be able to be written in dimensionless form in which dimensional quantities  $Q_{ref,i}$  do not exist as they are dependent on a choice of measure unit system. This form is presented below:

$$\tilde{F}(\{\dots, \tilde{Q}_{ref,i}, \dots\}; \{\dots, 1_{ref,i}, \dots\}; \{\dots, \tilde{\Pi}_{Q_{ref,j}}, \dots\}) = 0 \quad (4.22)$$

If we change the scale of quantities:  $Q_i$ , i.e.:

$$\frac{Q_i^M}{Q_i^N} = \tilde{k}_{Qi} = \frac{Q_{ref,i}^M}{Q_{ref,i}^N} = \frac{\tilde{Q}_{ref,i}^M Q_{ref,i}^M}{\tilde{Q}_{ref,i}^N Q_{ref,i}^N} \quad (4.23)$$

dimensionless quantities  $\tilde{Q}_{ref,i}$  do not change (i.e.  $\tilde{Q}_{ref,i}^M = \tilde{Q}_{ref,i}^N$ ). Dimensionless quantities  $\tilde{\Pi}_{Q_{ref,j}}$  are or may be changed. And actually they represent model similarity criteria of two phenomena of similar nature but in two different scales.

Relationship (4.22) can be written shorter:

$$\tilde{F}(\{\dots, \tilde{Q}_{ref,i}, \dots\}; \{\dots, \tilde{\Pi}_{Q_{ref,j}}, \dots\}) = 0 \quad (4.24)$$

Numbers  $\tilde{Q}_{ref,i}$  are in a specific case known numbers (constants) whereas variable dimensionless quantities (dependent or independent) of a specific case (physical phenomenon, process) are dimensionless numbers  $\tilde{\Pi}_{Q_{ref,j}}$ .

In some special case, when  $Q_{ref,i} = Q_i$ ,  $\tilde{Q}_{ref,i} = 1$ , the form of the relationship (4.24) takes an analogous form as in first case of deriving the theorem II.

## 5. Model similarity scales of physical phenomena

From dimensional analysis one can also derived certain general relationships, which can be compiled in the case of scales of physical quantities characterizing some physical phenomenon. Therefore, if a set of these physical quantities is denoted as  $(Q_p, Q_q, Q_r, Q_1, Q_2, \dots, Q_n)$ , where  $(Q_p, Q_q, Q_r)$  form dimensional base of phenomenon, i.e. dimensionally independent quantities, which contain dimensions of basic base  $M, L$  and  $T$  (e.g. in mechanical problems), the following relationships can be written:

$$[Q_i] = [Q_p]^{\alpha_i} [Q_q]^{\beta_i} [Q_r]^{\gamma_i} \quad [Q_n] = [Q_p]^{\alpha_n} [Q_q]^{\beta_n} [Q_r]^{\gamma_n} \quad (5.1)$$

$$\begin{aligned} \left( \frac{Q_i}{Q_p^{\alpha_i} Q_q^{\beta_i} Q_r^{\gamma_i}} \right)_M &= \tilde{\Pi}_{iM} = \tilde{\Pi}_{iN} = \left( \frac{Q_i}{Q_p^{\alpha_n} Q_q^{\beta_n} Q_r^{\gamma_n}} \right)_N \\ \left( \frac{Q_n}{Q_p^{\alpha_n} Q_q^{\beta_n} Q_r^{\gamma_n}} \right)_M &= \tilde{\Pi}_{nM} = \tilde{\Pi}_{nN} = \left( \frac{Q_n}{Q_p^{\alpha_n} Q_q^{\beta_n} Q_r^{\gamma_n}} \right)_N \end{aligned} \quad (5.2)$$



or

$$\frac{Q_{1M}}{Q_{1N}} = \bar{k}_{Q_1} = \frac{(Q_p^{\alpha_1} Q_q^{\beta_1} Q_r^{\gamma_1})_M}{(Q_p^{\alpha_1} Q_q^{\beta_1} Q_r^{\gamma_1})_N} = \bar{k}_{Q_p}^{\alpha_1} \bar{k}_{Q_q}^{\beta_1} \bar{k}_{Q_r}^{\gamma_1} \quad \frac{Q_{nM}}{Q_{nN}} = \bar{k}_{Q_n} = \frac{(Q_p^{\alpha_n} Q_q^{\beta_n} Q_r^{\gamma_n})_M}{(Q_p^{\alpha_n} Q_q^{\beta_n} Q_r^{\gamma_n})_N} = \bar{k}_{Q_p}^{\alpha_n} \bar{k}_{Q_q}^{\beta_n} \bar{k}_{Q_r}^{\gamma_n} \quad (5.3)$$

where:  $\bar{k}_{Q_p}, \bar{k}_{Q_q}, \bar{k}_{Q_r}$  - scales of dimensional base quantities;  $\bar{k}_{Q_1}, \dots, \bar{k}_{Q_n}$  - scales of the other dimensional dependent quantities.

Therefore, it is assumed that the dimensional base of some issue represent the following quantities: velocity  $v$ , length  $L$  and density  $\rho$  (or also reference quantities  $v_o, L_o, \rho_o$ ), then, transferring the results of measurements obtained from the model to the object in the natural scale, the following relations between scales of dimensional base  $\bar{k}_v, \bar{k}_L, \bar{k}_\rho$  and the scales of the other physical quantities characterizing that phenomenon should be used:

- the scale of actions  $\bar{k}_p$  :

$$[P] = [v]^2 [L]^2 [\rho]^1; \quad \bar{k}_p = \bar{k}_v^2 \bar{k}_L^2 \bar{k}_\rho \quad (5.4)$$

- the scale of pressures (stresses)  $\bar{k}_p$  :

$$[p] = [P]^1 [L]^{-2}; \quad \bar{k}_p = \bar{k}_v^2 \bar{k}_\rho \quad (5.5)$$

- the scale of time  $\bar{k}_t$  :

$$[t] = [v]^{-1} [L]^1; \quad \bar{k}_t = \bar{k}_v^{-1} \bar{k}_L \quad (5.6)$$

- the scale of frequency  $\bar{k}_f$  :

$$[f] = [v]^1 [L]^{-1}; \quad \bar{k}_f = 1 / \bar{k}_t = \bar{k}_v \bar{k}_L^{-1} \quad (5.7)$$

- the scale of mass density per unit length of element  $\bar{k}_m$  :

$$[m] = [L]^2 [\rho]^1; \quad \bar{k}_m = \bar{k}_L^2 \bar{k}_\rho \quad (5.8)$$

- the scale of moment of mass inertia density per unit length of element  $\bar{k}_{mb}$  :

$$[m_b] = [L]^4 [\rho]^1; \quad \bar{k}_{mb} = \bar{k}_L^4 \bar{k}_\rho \quad (5.9)$$

- the scale of longitudinal rigidity  $\bar{k}_{EA}$ , flexural rigidity  $\bar{k}_{EI}$  and torsional rigidity  $\bar{k}_{GIs}$  :

$$[EA] = [v]^2 [L]^2 [\rho]; \quad \bar{k}_{EA} = \bar{k}_v^2 \bar{k}_L^2 \bar{k}_\rho \quad (5.10)$$

$$[EI] = [v]^2 [L]^4 [\rho]; \quad \bar{k}_{EI} = \bar{k}_v^2 \bar{k}_L^4 \bar{k}_\rho \quad (5.11)$$

$$[GIs] = [v]^2 [L]^4 [\rho]; \quad \bar{k}_{GIs} = \bar{k}_v^2 \bar{k}_L^4 \bar{k}_\rho = \bar{k}_{EI} \quad (5.12)$$

## 6. Original generalized theorems II of dimensional analysis and model similarity of physical phenomena

Let a given physical phenomenon (process, event) describe different physical/mathematical relationships, and these relationships contain the following set of all physical quantities characteristic of this phenomenon:

$$\{S\} = \left\{ \{ \dots, Q_i, \dots \}; \{ \dots, \tilde{\Pi}_{Q,\alpha}, \dots \}; \{ \dots, C_\beta, \dots \}; \{ \dots, \tilde{C}_\gamma, \dots \} \right\} \quad (6.1)$$

where the subscripts:  $i = 1, 2, \dots, n$ ;  $\alpha = 1, 2, \dots, N_\alpha$ ;  $\beta = 1, 2, \dots, N_\beta$ ;  $\gamma = 1, 2, \dots, N_\gamma$

The set  $\{S\}$  contains the following subsets:

- $\{ \dots, Q_i, \dots \}$  – subset of dimensional quantities, which can be independent variables, dependent variables or parameters;
- $\{ \dots, \tilde{\Pi}_{Q,\alpha}, \dots \}$  – subset of dimensionless quantities, which can also be independent variables, dependent variables or parameters;
- $\{ \dots, C_\beta, \dots \}$  – subset of constant dimensional quantities;
- $\{ \dots, \tilde{C}_\gamma, \dots \}$ , – subset of constant dimensionless quantities (i.e. subset containing constant numbers).

Quantities  $Q_i$ ,  $\tilde{\Pi}_{Q,\alpha}$  and  $C_\beta$  depend on the physical phenomenon scale (may take different values at the natural scale and at the model scale).

For all of dimensional quantities  $Q_i$  and  $C_\beta$  of the problem analysed some reference quantities are assumed:

$$Q_i = \tilde{Q}_{ref,i} Q_{ref,i} \quad C_\beta = \tilde{C}_{ref,\beta} C_{ref,\beta} \quad (6.2)$$

Further we may assume that  $C_\beta = C_{ref,\beta}$ , so  $\tilde{C}_{ref,\beta} = 1$ . Let the physical/mathematical model of the physical phenomenon analysed describe a set of relationships comprising the functional operators  $F(\dots)$ , equation operators  $E(\dots)$  and inequality operators  $I(\dots)$  of the form:

$$F_p \left( \{ \dots, Q_i, \dots \}_p; \{ \dots, \tilde{\Pi}_{Q,\alpha}, \dots \}_p; \{ \dots, C_\beta, \dots \}_p; \{ \dots, \tilde{C}_\gamma, \dots \}_p \right) = O_p; \quad p = 1, 2, \dots, N_p \quad (6.3)$$

$$E_q \left( \{ \dots, Q_i, \dots \}_q; \{ \dots, \tilde{\Pi}_{Q,\alpha}, \dots \}_q; \{ \dots, C_\beta, \dots \}_q; \{ \dots, \tilde{C}_\gamma, \dots \}_q \right) = O_q; \quad q = 1, 2, \dots, N_q \quad (6.4)$$

$$I_r \left( \{ \dots, Q_i, \dots \}_r; \{ \dots, \tilde{\Pi}_{Q,\alpha}, \dots \}_r; \{ \dots, C_\beta, \dots \}_r; \{ \dots, \tilde{C}_\gamma, \dots \}_r \right) \leq O_r \text{ or } \geq O_r; \quad r = 1, 2, \dots, N_r \quad (6.5)$$

where  $\{S\}_p$ ,  $\{S\}_q$ ,  $\{S\}_r$  denote respective  $p$ ,  $q$ ,  $r$  subsets of the initial set  $\{S\}$ .

Assuming for dimensional quantities  $Q_i$  and  $C_\beta$  a dimensional base  $\{Q_{ref,1}, Q_{ref,2}, \dots, Q_{ref,k}\} = \{ \dots, Q_{ref,l}, \dots \}$ ;  $l = 1, 2, \dots, k$  which is made from chosen dimensional quantities  $Q_{ref,i}$  of the set  $\{ \dots, Q_{ref,i}, \dots \}$ , the following dimensionless quantities can be defined (com. p. 4.3):

$$\tilde{\Pi}_{Q_{ref,j}} = \tilde{Q}_{ref,j} \frac{Q_{ref,j}}{Q_{ref,1}^{a_{ref,1j}} Q_{ref,2}^{a_{ref,2j}} \dots Q_{ref,k}^{a_{ref,kj}}}; \quad (6.6)$$

$$j = k+1, k+2, \dots, n; n = k+1$$

$$\tilde{\Pi}_{Cref,\beta} = \frac{C_\beta}{Q_{ref,1}^{a_{ref,1\beta}} Q_{ref,2}^{a_{ref,2\beta}} \dots Q_{ref,k}^{a_{ref,k\beta}}} \quad (6.7)$$

Taking into account the basic principles of dimensional analysis and theory of similarity of physical phenomena presented previously (i.e. the principle of dimensional homogeneity and the objectivity of physical phenomena, that is, independence from mathematical relationships describing these phenomena in the unit system), the dimensional dependences (6.3), (6.4) and (6.5) constituting the physical/mathematical model of the given physical phenomenon, can be brought into the following dimensionless form:

$$\tilde{F}_p \left( \left\{ \dots, \tilde{Q}_{ref,l}, \dots \right\}_p; \left\{ \dots, \tilde{\Pi}_{Qref,j}, \dots \right\}_p; \left\{ \dots, \tilde{\Pi}_{Q,\alpha}, \dots \right\}_p; \left\{ \dots, \tilde{\Pi}_{Cref,\beta}, \dots \right\}_p; \left\{ \dots, \tilde{C}_\gamma, \dots \right\}_p \right) = 0_p \quad (6.8)$$

$$\tilde{E}_q \left( \left\{ \dots, \tilde{Q}_{ref,l}, \dots \right\}_q; \left\{ \dots, \tilde{\Pi}_{Qref,j}, \dots \right\}_q; \left\{ \dots, \tilde{\Pi}_{Q,\alpha}, \dots \right\}_q; \left\{ \dots, \tilde{\Pi}_{Cref,\beta}, \dots \right\}_q; \left\{ \dots, \tilde{C}_\gamma, \dots \right\}_q \right) = 0_q \quad (6.9)$$

$$\tilde{I}_r \left( \left\{ \dots, \tilde{Q}_{ref,l}, \dots \right\}_r; \left\{ \dots, \tilde{\Pi}_{Qref,j}, \dots \right\}_r; \left\{ \dots, \tilde{\Pi}_{Q,\alpha}, \dots \right\}_r; \left\{ \dots, \tilde{\Pi}_{Cref,\beta}, \dots \right\}_r; \left\{ \dots, \tilde{C}_\gamma, \dots \right\}_r \right) = 0_r \quad (6.10)$$

Changing the scale of the physical phenomenon, the values of numerical quantities  $\tilde{Q}_{ref,l}$  and  $\tilde{C}_\gamma$  do not change. All the other dimensionless quantities  $\tilde{\Pi}$ , i.e.  $\tilde{\Pi}_{Qref,j}$ ,  $\tilde{\Pi}_{Q,\alpha}$  and  $\tilde{\Pi}_{Cref,\beta}$  depend on the scale of the given physical phenomenon.

Taking into account the above considerations, generalized  $\Pi$ -theorems of dimensional analysis and physical phenomena model similarity can be formulated as follows:

1. A physical/mathematical model of some physical phenomenon can be described by a system of independent mathematical relationships in dimensionless form containing dimensionless mathematical operators of functions:  $\tilde{F}_p(\dots)$ , equations  $\tilde{E}_q(\dots)$ , inequalities  $\tilde{I}_r(\dots)$  and dimensionless quantities characterizing this phenomenon:  $\tilde{Q}_{ref,l}$ ,  $\tilde{\Pi}_{Qref,j}$ ,  $\tilde{\Pi}_{Q,\alpha}$ ,  $\tilde{\Pi}_{Cref,\beta}$ ,  $\tilde{C}_\gamma$ ;  $l = 1, 2, \dots, k$ ;  $j = k+1, k+2, \dots, n$ ;  $\alpha = 1, 2, \dots, N_\alpha$ ;  $\beta = 1, 2, \dots, N_\beta$ ;  $\gamma = 1, 2, \dots, N_\gamma$ . Dimensionless quantities  $\tilde{Q}_{ref,l}$ ,  $\tilde{C}_\gamma$  are numbers independent on a scale of some physical phenomenon, whereas dimensionless quantities  $\tilde{\Pi}_{Qref,j}$ ,  $\tilde{\Pi}_{Q,\alpha}$ ,  $\tilde{\Pi}_{Cref,\beta}$  are numbers dependent on the scale of this phenomenon. The dimensionless quantities  $\tilde{\Pi}_{Qref,j}$  and  $\tilde{\Pi}_{Cref,\beta}$  connected with the dimensional quantities of physical phenomenon  $Q_j$  and  $C_\beta$  are  $k$  less than all the dimensional quantities  $Q_i$  and  $C_\beta$ , where  $k$  is the number of elements of the dimensional base of the physical phenomenon.
2. Two phenomena with similar physical nature performed on different scales (e.g. natural scale  $N$  and model scale  $M$ ) are similar if the set of relationships describing the physical/mathematical model of this phenomenon in dimensionless form (6.8), (6.9), (6.10) is the same. The numbers  $\tilde{\Pi}_{Qref,j}$ ,  $\tilde{\Pi}_{Q,\alpha}$ ,  $\tilde{\Pi}_{Cref,\beta}$  dependent on the scale of the phenomenon are similarity numbers (criteria) of the model similarity of this phenomenon. These numbers, for phenomena executed on two scales, should be equal. At partial model similarity, this satisfies only the equality of the most important criterial numbers, which have the greatest impact on the results of the studied phenomenon. This case is the most common in practice.
3. In the case of cause end effect phenomena, among all dimensionless quantities occurring in relationships (6.8), (6.9), (6.10), we can distinguish quantities connected with input

IN, system (object) O and output OU (or several inputs, subsystems and outputs). The essence of model investigations is to measure in model tests the values of output OU of dimensionless quantities at fulfilment equalities of the greatest number as possible of the other criterial numbers, and at assuming that the physical nature of the phenomenon executed in two scales is similar. Dimensionless quantities of output can be transferred from the model scale to the natural scale (or other) since they represent similarity criteria of the investigated physical phenomenon.

## **7. The nature of physical phenomena occurring in mechanics of continuous or discrete material mediums**

### 7.1. Concept of an input/output physical system and its physical/mathematical model. Block diagram of a system

Material mediums in mechanics usually are divided into solid body (stiff or deformable) and fluids (liquids and gases). Physical phenomena in mechanics have a cause and effect character.

If for the processes or phenomena related to some material system (object) there is a cause and effect relationship (or relationships), then the block of data related to the cause is called the input (IN), the block of data related to the material system (object), which is the subject of input influence is called the system, characteristic object, or simply object (O), and the third block of data related to the effect of that influence is called the output (OU). In mechanics of material mediums is often called the input action, load or force acting on the system; the system (object) may be called the system, structure, construction, etc. and the output is called response or reaction of the system. Every set of these three blocks of data with one input, one object and one output we will call a simple system (comp. Fig. 2)

If several inputs IN act on the system O and there are also several outputs OU, we call such a system a complex system with several inputs and several outputs. Likewise, each of the series connections, parallel connections, or series-parallel connections of the simple systems create complex series, parallel or series-parallel systems. It is also possible to make complex mixed systems (e.g. a series system including subsystems with a large number of inputs and outputs). Issues of mechanics of material mediums can be classified just as input/output complex mixed systems.

Input and output quantities are sometimes dependent on each other. When system output quantities can influence system input quantities, then that system is called a system with feedback. The aerodynamic feedback which occurs between building vibrations and wind actions on a building is an example of such feedback. Building vibrations change the character of the air flow around the building, and thereby change the distribution of wind pressures on the walls of the building.

Examples of different complex input/output systems are shown schematically in Fig. 3.

For example, the system shown in Fig. 3c can be interpreted as follows: subset O1 is the domain of ground foundations adjacent to building foundations; input IN1 represents vibrations of ground foundations on the part of the outer surface of this domain from the

vibration source direction, which can be seismic or para-seismic excitations; output OU1 represents accelerations (displacements, velocities) of building foundations, which in turn represent cinematic excitation of the building itself, i.e. object O; input IN2 is for example the wind velocity field in front of the building, in the part of the outer surface of the air domain which is adjacent to the building, so subset O2; output OU2 represents the wind pressure field on the walls of the building, which in turn is wind action on the building itself.

The description of the relationships occurring in subsystems of the whole input/output system in mathematical formal “language” we call the mathematical model of that system.

The presentation of a system, actions (loads) on this system, and relationships occurring between them, using a set of some conventional graphic symbols in structural mechanics is called the static scheme of the system (if time is not significant) or the dynamic scheme of the system (if time is significant). If in consideration it is possible to omit the feature of the system called inertia, the dynamic scheme of the system is called the rheology scheme.

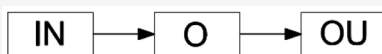


Fig. 2. Block diagram of a simple system (with one input and one output)

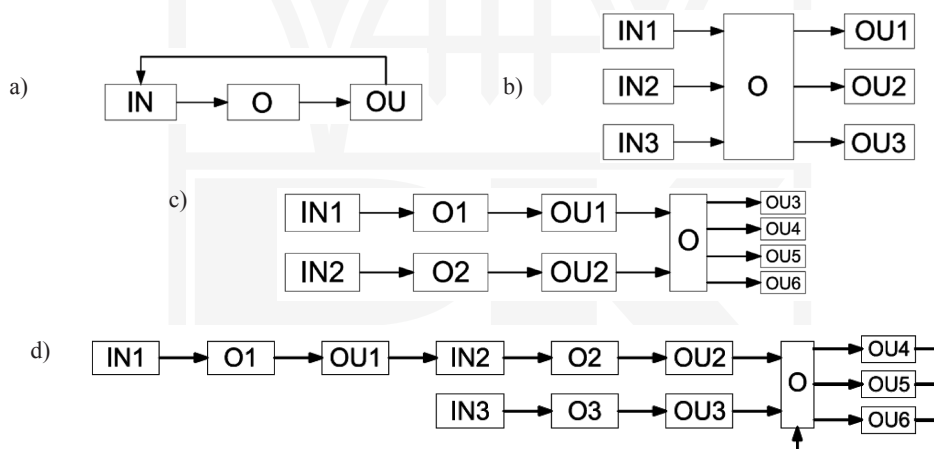


Fig. 3. Examples of block diagrams of complex input/output systems: a) simple system with feedback, b) system with several inputs and outputs, c) parallel system with several inputs and outputs, d) series-parallel system with feedback and several inputs and outputs

## 7.2. Initial and temporary (final) state of the system.

Space-time variables used to describe the object's movement

The position of particular points of an object in the initial state ( $t_o$ ) and a temporary state ( $t$ ) is described by coordinates of radius vectors of these points:  $\mathbf{r}_{oi}(t_o)$  and  $\mathbf{r}_i(t)$ , i.e.

$\{\dots, x_{oi}, \dots\}$  and  $\{\dots, x_i(t), \dots\}$ , where  $i$  – is the subscript which identifies any  $i$ -th point of the system. These coordinates can be defined in the global coordinate system for the whole system  $OXYZ$  and/or in local coordinate systems  $O_e x_e y_e z_e$  of the various parts of the system, where  $e$  – the subscript which identify any  $e$ -th element of the system. All of the geometrical quantities characterizing the transition from the system's initial state to a temporary state (displacements, deformations, velocities, accelerations, etc.) can be determined as a function of the initial coordinates  $\{\dots, x_{oi}, \dots\}$  or of temporary coordinates  $\{\dots, x_i(t), \dots\}$  and time  $t$ . In the first case we talk about a material description (Lagrange's substantial description) of the system movement, and in the second case we talk about a spatial description (Euler's description). It is similar with the other variables or parameters of particular inputs  $IN_\alpha$  of objects  $O_\beta$  and outputs  $OU_\gamma$ , where  $\alpha, \beta, \gamma$  – subscripts identifying particular inputs, objects and outputs of the whole system.

### 7.3. Types of relationships describing some mechanical phenomenon

In descriptions of some mechanical phenomenon we can distinguish the following groups of geometrical and physical relationships, which we express as mathematical relationships:

- Geometrical relationships describing initial geometry (configuration) of a single system or particular subsystems of the whole system (e.g. boundary surface equations of solid body, axle geometry equations of particular bars of bar system);
- Relationships for quantities related to the restrictions imposed on the system/subsystems, which result from the existence of different type of external/internal constraints of the system/subsystems, which restrain their movement/deformations. These can be kinematic constraints, mechanical constraints or out of mechanical constraints;
- Relationships resulting from initial conditions with respect to excitations (actions) kinematic, mechanical, out of mechanical acting on the system/subsystems;
- Relationships arising from geometry and mechanical laws, connecting quantities which describe the transition of system/subsystems from the initial state (initial configuration) to a temporary/final state (temporary/final configuration);
- Relationships related to imposing different types of restrictions on the system/subsystems connected with their serviceability (serviceability conditions) and safety (limit conditions).

These groups of relationships are associated with specific groups of geometrical and physical quantities (dimensional or dimensionless variables/parameters) dependent on the scale of the phenomenon, dimensional or dimensionless constants independent of the scale of the phenomenon), which in turn represent different subsets  $\{S\}_s$  of the initial set  $\{S\}$  of all the quantities characterizing this mechanical phenomenon.

The essence of model investigations of mechanical phenomena consists of performing respective investigations (tests) of that phenomenon at a smaller scale, on the fulfilment of model similarity criteria of that phenomenon, and on measurement of dimensionless output quantities of particular subsystems or the system as the whole and the transition of the same values of these quantities to the mechanical phenomenon at the natural scale (or another scale).

### 8. Example – vibration of a system with one degree of freedom with mechanical and kinematic excitation

The scheme of the system is shown in the Fig. 4.

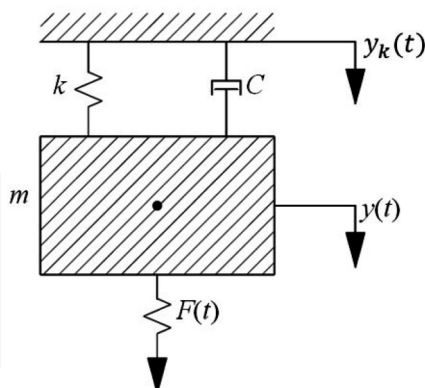


Fig. 4. Scheme of a system with one degree of freedom with kinematic  $y_k(t)$  and mechanical excitation  $F(t)$

#### 8.1. Basic denotations and relationships

- The system parameters:  $m$  – mass;  $c$  - damping,  $k$ – rigidity
- Kinematic excitation:

$$y_k(t) = f_{y_k}(t; Y_k, \sigma_{y_k}) \quad (8.1)$$

Kinematic excitation is a stochastic process of parameters:  $Y_k$  (amplitude of excitation) and  $\sigma_{y_k}$  (standard deviation or root-mean-square value of excitation).

- Mechanical excitation (action):

$$F(t) = mg + P_0 \sin(\theta t + \varphi) \quad (8.2)$$

where:  $mg$  – gravity force ( $g$  – acceleration of gravity);  $P_0$  – amplitude of harmonic excitation;  $\theta = 2\pi f = \frac{2\pi}{T}$  – circular frequency of excitation ( $f$  – frequency of excitation,  $T$  – period of excitation);  $\varphi$  – shift angle in radians.

- Static displacement of the system:

$$y_{st} = \frac{mg}{k} = \frac{g}{\omega^2} = \frac{gT_0^2}{4\pi^2} \quad (8.3)$$

- Absolute displacement  $y(t)$ , dynamic displacement  $y_{dyn}(t)$  and relative displacement  $y_{rel}(t)$  of the system:

$$y(t) = y_{st} + y_{dyn} \quad (8.4)$$

$$y_{rel}(t) = y_{dyn}(t) - y_k(t) \quad (8.5)$$

$$y(t) = y_{st} + y_{rel}(t) + y_k(t) \quad (8.6)$$

- System movement equation resulting from the laws of mechanics:

$$m \frac{d^2 y(t)}{dt^2} + \left( \frac{dy(t)}{dt} - \frac{dy_k(t)}{dt} \right) + k(y(t) - y_k(t)) = mg + P_o \sin(\theta t + \varphi) \quad (8.7)$$

- Additional denotations:

1. Circular frequency of free vibration  $\omega$  ( $f_o$  – frequency of free vibration,  $T_o$  – period of free vibration)

$$\omega = \sqrt{\frac{k}{m}} = 2\pi f_o = \frac{2\pi}{T_o} \quad (8.8)$$

2. Dimensionless parameter of damping (damping ratio)  $\gamma$ :

$$\frac{c}{m} = 2\gamma\omega = 2\gamma\sqrt{\frac{k}{m}}; \quad \gamma = \frac{1}{2} \frac{c}{\sqrt{mk}} \quad (8.9)$$

- Different form of system motion equation:

$$\frac{d^2 y_{rel}(t)}{dt^2} + 2\gamma\omega \frac{dy_{rel}(t)}{dt} + \omega^2 y_{rel}(t) = -\frac{d^2 y_k(t)}{dt^2} + g + \frac{P_o}{m} \sin(\theta t + \varphi) \quad (8.10)$$

- Initial conditions in time instant  $t_o=0$

$$y_{rel}(0) = y_o; \quad \left. \frac{dy_{rel}}{dt} \right|_{t_o=0} = \frac{dy_{rel}(0)}{dt} = v_o \quad (8.11)$$

- Limits imposed on the system arising from serviceability and safety conditions:

$$y_{st} = \frac{gT_o^2}{4\pi^2} \leq y_{st,lim} \quad (8.12)$$

$$Y_{dyn} = y_{dyn,max} \leq y_{dyn,lim} \quad (8.13)$$

$$R(t) = 2\gamma\omega \frac{dy_{rel}(t)}{dt} + \omega^2 y_{rel}(t) \leq R_{lim} \quad (8.14)$$



## 8.2. Main set and subsets of dimensional and dimensionless quantities characterizing the problem analysed

- Primary set:

$$\{S\} = \left\{ y_{rel}, t, 2, \gamma, \omega = \frac{2\pi}{T_o}, y_k, Y_k, \sigma_{yk}, g, P_o, m, \theta = \frac{2\pi}{T}, \varphi, \nu_o, \nu_o, y_{st,lim}, Y_{dyn}, y_{dyn,lim}, R_{lim} \right\} \quad (8.15)$$

- Subset of variables (dependent and independent) and the dimensional parameters dependent on a phenomenon scale:

$$\{S\}_1 = \left\{ y_{rel}, t, \omega = \frac{2\pi}{T_o}, y_k, Y_k, \sigma_{yk}, P_o, m, \theta = \frac{2\pi}{T}, \nu_o, \nu_o, y_{st,lim}, Y_{dyn}, y_{dyn,lim}, R_{lim} \right\} \quad (8.16)$$

- Subset of dimensionless parameters dependent on a phenomenon scale:

$$\{S\}_2 = \{\gamma = \tilde{\gamma}\} \quad (8.17)$$

- Subset of dimensionless parameters independent on a phenomenon scale

$$\{S\}_3 = \{\varphi = \tilde{\varphi}\} \quad (8.18)$$

- Subset of constant dimensional quantities:

$$\{S\}_4 = \{g\} \quad (8.19)$$

- Subset of dimensionless (numerical) constants:

$$\{S\}_5 = \{2\} \text{ or } \{S\}_5 = \{2, 2\pi, 4\pi^2\} \quad (8.20)$$

dependent on the type of parameters that are used:  $\omega$  and  $\theta$  or  $T_o$  and  $T$ .

## 8.3. Dimensional analysis of the issue and the dimensionless forms of the relationships describing the physical/mathematical model of this issue

### 8.3.1. The basic reference base

Let the reference base of this issue represent the set of the following dimensional reference quantities:

$$\{B_{ref}\} = \{Y_{rel,ref}, T_{ref}, M_{ref}\} \quad (8.21)$$

where:  $Y_{rel,ref}, T_{ref}, M_{ref}$  – reference quantities connected with length, time and mass.

The dimensionless reference quantities and mathematical operators (functional, differential) corresponding to this base are as follows:

$$y_{rel} = \check{y}_{rel,ref} Y_{rel,ref} \quad (8.22) \quad t = \check{t}_{ref} T_{ref} \quad (8.23)$$

$$y_{rel}(t) = Y_{rel,ref} \frac{y_{rel}(\check{t}_{ref} T_{ref})}{Y_{rel,ref}} = Y_{rel,ref} \check{y}_{rel,ref}(\check{t}_{ref}) \quad (8.24)$$

$$d(y_{rel}(t)) = d(Y_{rel,ref} \check{y}_{rel,ref}(\check{t}_{ref})) = Y_{rel,ref} d(\check{y}_{rel,ref}(\check{t}_{ref})) \quad (8.25)$$

$$dt = d(\check{t}_{ref} T_{ref}) = T_{ref} d\check{t}_{ref} \quad (8.26) \quad \frac{d(y_{rel}(t))}{dt} = \frac{Y_{rel,ref}}{T_{ref}} \frac{d(\check{y}_{rel,ref}(\check{t}_{ref}))}{d\check{t}_{ref}} \quad (8.27)$$

$$\frac{d^2(y_{rel}(t))}{dt^2} = \frac{d}{dt} \left( \frac{d(y_{rel}(t))}{dt} \right) = \frac{Y_{rel,ref}}{T_{ref}^2} \frac{d^2(\check{y}_{rel,ref}(\check{t}_{ref}))}{d\check{t}_{ref}^2} \quad (8.28)$$

$$\omega = \frac{2\pi}{T_o} = \frac{2\pi}{\check{T}_{o,ref} T_{ref}} \quad (8.29)$$

$$y_k(t; Y_k, \sigma_{yk}) = Y_{rel,ref} \frac{y_k(\check{t}_{ref} T_{ref}; \check{Y}_{k,ref} Y_{rel,ref}; \check{\sigma}_{yk,ref} Y_{rel,ref})}{Y_{rel,ref}} = Y_{rel,ref} \check{y}_{k,ref}(\check{t}_{ref}; \check{Y}_{k,ref}; \check{\sigma}_{yk,ref}) \quad (8.30)$$

$$\frac{d^2(y_k(t); Y_k, \sigma_{yk})}{dt^2} = \frac{Y_{rel,ref}}{T_{ref}^2} \frac{d^2(\check{y}_{k,ref}(\check{t}_{ref}; \check{Y}_{k,ref}; \check{\sigma}_{yk,ref}))}{d\check{t}_{ref}^2} \quad (8.31)$$

$$g = \check{g}_{ref} \frac{Y_{rel,ref}}{T_{ref}^2} \quad (8.32) \quad P_o = \check{P}_{o,ref} \frac{Y_{rel,ref} M_{ref}}{T_{ref}^2} \quad (8.33)$$

$$m = \check{m}_{ref} M_{ref} \quad (8.34) \quad \theta = \frac{2\pi}{T} = \frac{2\pi}{\check{T}_{ref} T_{ref}} \quad (8.35)$$

$$y_o = y_{rel}(0) = \check{y}_{rel,ref}(0) Y_{rel,ref} \quad (8.36) \quad v_o = \frac{dy_{rel}(0)}{dt} = \frac{Y_{rel,ref}}{T_{ref}} \frac{d\check{y}_{rel,ref}(0)}{d\check{t}_{ref}} \quad (8.37)$$

$$y_{st,lim} = \check{y}_{st,lim,ref} Y_{rel,ref} \quad (8.38) \quad Y_{dyn} = \check{Y}_{dyn,ref} Y_{rel,ref} \quad (8.39)$$

$$y_{dyn,lim} = \check{y}_{dyn,lim,ref} Y_{ref,ref} \quad (8.40) \quad R_{lim} = R_{lim,ref} \frac{Y_{rel,ref}}{T_{ref}^2} \quad (8.41)$$

The relationships which describe the physical/mathematical model of the mechanical phenomenon analysed are written in dimensionless form as follows:

$$\begin{aligned} & \frac{Y_{ref}}{T_{ref}^2} \frac{d^2 \check{y}_{rel,ref}(\check{t}_{ref})}{d\check{t}_{ref}^2} + 2\check{\gamma} \frac{2\pi}{\check{T}_{o,ref} T_{ref}} \frac{Y_{ref}}{T_{ref}} \frac{d\check{y}_{rel,ref}}{d\check{t}_{ref}} + \frac{4\pi^2}{(\check{T}_{o,ref} T_{ref})^2} Y_{ref} \check{y}_{rel,ref}(\check{t}_{ref}) = \\ & = -\frac{Y_{ref}}{T_{ref}^2} \frac{d^2 \check{y}_{k,ref}(\check{t}_{ref}; \check{Y}_{k,ref}, \check{\sigma}_{yk,ref})}{d\check{t}_{ref}^2} + \check{g}_{ref} \frac{Y_{ref}}{T_{ref}^2} + \frac{\check{P}_{o,ref}}{\check{m}_{ref} M_{ref}} \frac{M_{ref} Y_{ref}}{T_{ref}^2} \sin\left(\frac{2\pi}{\check{T}_{ref} T_{ref}} \check{t}_{ref} T_{ref} + \check{\varphi}\right) \end{aligned} \quad (8.42)$$

or after dividing the equation by  $\frac{Y_{ref}}{T_{ref}^2}$ :

$$\begin{aligned} & \frac{d^2 \check{y}_{rel,ref}(\check{t}_{ref})}{d\check{t}_{ref}^2} + 2\check{\gamma} \frac{2\pi}{\check{T}_{o,ref}} \frac{d\check{y}_{rel,ref}(\check{t}_{ref})}{d\check{t}_{ref}} + \frac{4\pi^2}{\check{T}_{o,ref}^2} \check{y}_{rel,ref}(\check{t}_{ref}) = \\ & = -\frac{d^2 \check{y}_{k,ref}(\check{t}_{ref}; \check{Y}_{k,ref}, \check{\sigma}_{yk,ref})}{d\check{t}_{ref}^2} + \check{g}_{ref} + \frac{\check{P}_{o,ref}}{\check{m}_{ref}} \sin\left(\frac{2\pi}{\check{T}_{ref}} \check{t}_{ref} + \check{\varphi}\right) \end{aligned} \quad (8.43)$$

$$\check{y}_{rel,ref}(0) = \frac{y_o}{Y_{ref}} \quad (8.44) \quad \frac{d\check{y}_{rel,ref}(0)}{d\check{t}_{ref}} = \frac{v_o T_{ref}}{Y_{ref}} \quad (8.45)$$

$$\check{g}_{ref} \frac{Y_{ref}}{T_{ref}^2} \frac{\check{T}_{o,ref}^2 T_{ref}^2}{4\pi^2} \leq \check{y}_{st,lim,ref} Y_{ref} \quad (8.46) \quad \text{or:} \quad \check{g}_{ref} \frac{\check{T}_{o,ref}^2}{4\pi^2} \leq \check{y}_{st,lim,ref} \quad (8.47)$$

$$\check{Y}_{dyn,ref} Y_{ref} \leq \check{y}_{dyn,lim,ref} Y_{ref} \quad (8.48) \quad \text{or:} \quad \check{Y}_{dyn,ref} \leq \check{y}_{dyn,lim,ref} \quad (8.49)$$

$$2\check{\gamma} \frac{2\pi}{\check{T}_{o,ref} T_{ref}} \frac{Y_{ref}}{T_{ref}} \frac{d\check{y}_{rel,ref}(\check{t}_{ref})}{d\check{t}_{ref}} + \frac{4\pi^2}{\check{T}_{o,ref}^2 T_{ref}^2} Y_{ref} \check{y}_{rel,ref}(\check{t}_{ref}) \leq \frac{Y_{ref}}{T_{ref}^2} \check{R}_{lim,ref} \quad (8.50)$$

or

$$2\check{\gamma} \frac{2\pi}{\check{T}_{o,ref}} \frac{d\check{y}_{rel,ref}(\check{t}_{ref})}{d\check{t}_{ref}} + \frac{4\pi^2}{\check{T}_{o,ref}} \check{y}_{rel,ref}(\check{t}_{ref}) \leq \check{R}_{lim,ref} \quad (8.51)$$

When the scale of the mechanical phenomenon is changed, i.e. when the corresponding scales are:

$$\tilde{k}_{yrel} = \tilde{k}_{Yrel,ref}; \tilde{k}_t = \tilde{k}_{Tref}; \tilde{k}_m = \tilde{k}_{Mref} \quad (8.52)$$

dimensionless quantities:  $\tilde{y}_{rel,ref}, \tilde{t}_{ref}, \tilde{m}_{ref}$  are the same at the natural scale ( $N$ ) as well as at the model scale ( $M$ ), so that they are not dependent on the scale of the phenomenon.

Moreover, in the above relationships the following dimensionless quantities  $\tilde{\Pi}_{ref,j}$ ,  $j = 1, 2, \dots, 15$  occur dependent on the scale of the mechanical phenomenon, which constitute the model similarity criteria (conditions, numbers) of that phenomenon:

$$\tilde{\gamma} = \frac{c}{2\sqrt{km}} = \tilde{\Pi}_{ref,1} \quad (8.53) \quad T_{o,ref} = \frac{2\pi}{\sqrt{\frac{k}{m}T_{ref}}} = \tilde{\Pi}_{ref,2} \quad (8.54)$$

$$\tilde{y}_{k,ref} = \frac{y_k}{Y_{rel,ref}} = \tilde{\Pi}_{ref,3} \quad (8.55) \quad \tilde{Y}_k = \frac{Y_k}{Y_{rel,ref}} = \tilde{\Pi}_{ref,4} \quad (8.56)$$

$$\tilde{\sigma}_{yk,ref} = \frac{\sigma_{yk}}{Y_{rel,ref}} = \tilde{\Pi}_{ref,5} \quad (8.57) \quad \tilde{g}_{ref} = \frac{gT_{ref}^2}{Y_{rel,ref}} = \tilde{\Pi}_{ref,6} \quad (8.58)$$

$$\tilde{P}_{o,ref} = \frac{P_o T_{ref}^2}{M_{ref} Y_{rel,ref}} = \tilde{\Pi}_{ref,7} \quad (8.59) \quad \tilde{T}_{ref} = \frac{2\pi}{\theta_{T_{ref}}} = \tilde{\Pi}_{ref,8} \quad (8.60)$$

$$\tilde{\varphi} = \tilde{\Pi}_{ref,9} \quad (8.61) \quad \tilde{y}_{rel,ref}(0) = \frac{y_o}{Y_{rel,ref}} = \tilde{\Pi}_{ref,10} \quad (8.62)$$

$$\frac{d\tilde{y}_{rel,ref}(0)}{dt_{ref}} = \frac{v_o T_{ref}}{T_{rel,ref}} = \tilde{\Pi}_{ref,11} \quad (8.63) \quad \tilde{y}_{st,lim,ref} = \frac{y_{st,lim}}{Y_{rel,ref}} = \tilde{\Pi}_{ref,12} \quad (8.64)$$

$$\tilde{Y}_{dyn,ref} = \frac{Y_{dyn}}{Y_{rel,ref}} = \tilde{\Pi}_{ref,13} \quad (8.65) \quad \tilde{y}_{dyn,lim,ref} = \frac{y_{dyn,lim}}{Y_{rel,ref}} = \tilde{\Pi}_{ref,14} \quad (8.66)$$

$$\tilde{R}_{lim,ref} = \frac{R_{dop} T_{ref}^2}{Y_{rel,ref}} = \tilde{\Pi}_{ref,15} \quad (8.67)$$

Thus, there are fifteen dimensionless numbers in total.

Moreover, in relationships describing the mechanical issue analysed, constant numbers appear, e.g.  $2, 2\Pi, 4\Pi^2$ .

Generally, with the exception of constant numbers, the number of dimensional and dimensionless output quantities characterizing the mechanical phenomenon is eighteen, so three more than numbers  $\tilde{\Pi}_{ref,j}$ , i.e. about as much as the size of the dimensional base of this issue.

## 8.3.2. Other reference base

One may consider now the case of another reference base of the issue which consists of the following three dimensional reference parameters:

$$\{B_{ref}^*\} = \{B\} = \{\omega, g, m\} \quad (8.68)$$

The respective relationships in this case are as follows:

$$y_{rel} = \tilde{y}_{rel} \frac{g}{\omega^2} \quad (8.69) \quad t = \tilde{t} \frac{1}{\omega} \quad (8.70)$$

$$\frac{dy_{rel}(t)}{dt} = \frac{g}{\omega} \frac{d\tilde{y}_{rel}(\tilde{t})}{d\tilde{t}} \quad (8.71) \quad \frac{d^2 y_{rel}(t)}{dt^2} = g \frac{d^2 \tilde{y}_{rel}(\tilde{t})}{d\tilde{t}^2} \quad (8.72)$$

$$\gamma = \tilde{\gamma} \quad (8.73) \quad y_k(t; Y_k, \sigma_{yk}) = \frac{g}{\omega^2} \tilde{y}_k(\tilde{t}; \tilde{Y}_k, \tilde{\sigma}_{yk}) \quad (8.74)$$

$$Y_k = \tilde{Y}_k \frac{g}{\omega^2} \quad (8.75) \quad \sigma_{yk} = \tilde{\sigma}_{yk} \frac{g}{\omega^2} \quad (8.76)$$

$$\frac{d^2 y_k(t; Y_k, \sigma_{yk})}{dt^2} = g \frac{d^2 \tilde{y}_k(\tilde{t}; \tilde{Y}_k, \tilde{\sigma}_{yk})}{d\tilde{t}^2} \quad (8.77) \quad P_o = \tilde{P}_o mg \quad (8.78)$$

$$\theta = \tilde{\theta} \omega \quad (8.79) \quad \varphi = \tilde{\varphi} \quad (8.80)$$

$$y_o = \tilde{y}_o \frac{g}{\omega^2} \quad (8.81) \quad v_o = \tilde{v}_o \frac{g}{\omega} \quad (8.82)$$

$$y_{st,lim} = \tilde{y}_{st,lim} \frac{g}{\omega^2} \quad (8.83) \quad Y_{dyn} = \tilde{Y}_{dyn} \frac{g}{\omega^2} \quad (8.84)$$

$$y_{dyn,lim} = \tilde{y}_{dyn,lim} \frac{g}{\omega^2} \quad (8.85) \quad R_{lim} = \tilde{R}_{lim} g \quad (8.86)$$

$$\begin{aligned} & g \frac{d^2 \tilde{y}_{rel}(\tilde{t})}{d\tilde{t}^2} + 2\tilde{\gamma} \omega \frac{g}{\omega} \frac{d\tilde{y}_{rel}(\tilde{t})}{d\tilde{t}} + \omega^2 \frac{g}{\omega^2} \tilde{y}_{rel}(\tilde{t}) = \\ & = -g \frac{d^2 \tilde{y}_k(\tilde{t}; \tilde{Y}_k, \tilde{\sigma}_{yk})}{d\tilde{t}^2} + g + \frac{\tilde{P}_o mg}{m} \sin\left(\tilde{\theta} \omega \frac{1}{\omega} \tilde{t} + \tilde{\varphi}\right) \end{aligned} \quad (8.87)$$

or after dividing equation by  $g$ :

$$\frac{d^2 \tilde{y}_{rel}(\tilde{t})}{d\tilde{t}^2} + 2\tilde{\gamma} \frac{d\tilde{y}_{rel}(\tilde{t})}{d\tilde{t}} + y_{rel}(\tilde{t}) = -\frac{d^2 \tilde{y}_k(\tilde{t}, \tilde{Y}_k, \tilde{\sigma}_{y_k})}{d\tilde{t}^2} + 1 + \tilde{P}_o \sin(\tilde{\theta}\tilde{t} + \tilde{\varphi}) \quad (8.88)$$

$$\tilde{y}_o = \frac{y_o \omega^2}{g} \quad (8.89) \quad \tilde{v}_o = \frac{v_o \omega}{g} \quad (8.90)$$

$$1 \leq \tilde{y}_{st,lim} = \frac{y_{st,lim} \omega^2}{g} \quad (8.91) \quad \tilde{Y}_{dyn} = \frac{Y_{dyn} \omega^2}{g} \leq \tilde{y}_{dyn,lim} = \frac{y_{dyn,lim} \omega^2}{g} \quad (8.92)$$

$$2\tilde{\gamma} \frac{d\tilde{y}_{rel}(\tilde{t})}{d\tilde{t}} + \tilde{y}_{rel}(\tilde{t}) \leq \tilde{R}_{lim} = \frac{R_{lim}}{g} \quad (8.93)$$

In relationships (8.88) and (8.93)  $j = 15$  occurs following the dimensionless numbers  $\tilde{\Pi}_j$ , which constitute the similarity criteria of the case analysed at another dimensional base:

$$\tilde{y}_{rel} = \frac{y_{rel} \omega^2}{g} = \tilde{\Pi}_1 \quad (8.94) \quad \tilde{t} = \omega t = \tilde{\Pi}_2 \quad (8.95)$$

$$\tilde{\gamma} = \frac{c}{2\sqrt{km}} = \tilde{\Pi}_{ref,1} = \tilde{\Pi}_3 \quad (8.96) \quad \tilde{y}_k = \frac{y_k \omega^2}{g} = \tilde{\Pi}_4 \quad (8.97)$$

$$\tilde{Y}_k = \frac{Y_k \omega^2}{g} = \tilde{\Pi}_5 \quad (8.98) \quad \tilde{\sigma}_{y_k} = \frac{\sigma_{y_k} \omega^2}{g} = \tilde{\Pi}_6 \quad (8.99)$$

$$\tilde{P}_o = \frac{P_o}{mg} = \tilde{\Pi}_7 \quad (8.100) \quad \tilde{\theta} = \frac{\theta}{\omega} = \tilde{\Pi}_8 \quad (8.101)$$

$$\tilde{\varphi} = \varphi = \Pi_{ref,9} = \tilde{\Pi}_9 \quad (8.102) \quad \tilde{y}_o = \frac{y_o \omega^2}{g} = \tilde{\Pi}_{10} \quad (8.103)$$

$$\tilde{v}_o = \frac{v_o \omega}{g} = \tilde{\Pi}_{11} \quad (8.104) \quad \tilde{y}_{st,lim} = \frac{y_{st,lim} \omega^2}{g} = \tilde{\Pi}_{12} \quad (8.105)$$

$$\tilde{Y}_{dyn} = \frac{Y_{dyn} \omega^2}{g} = \tilde{\Pi}_{13} \quad (8.106) \quad \tilde{y}_{dyn,lim} = \frac{y_{dyn,lim} \omega^2}{g} = \tilde{\Pi}_{14} \quad (8.107)$$

$$\check{R}_{lim} = \frac{R_{lim}}{g} = \check{\Pi}_{15} \quad (8.108)$$

They are, as can be seen, similarity criteria numbers different from the ones in the first case. Moreover, in the dimensionless relationships of the physical/mathematical model of that issue another set of numerical constants  $\{2,1\}$  appear. The form of these relationships is simpler in the second case.

## 9. Conclusions

The basic principles and theorems of dimensional analysis and theory of model similarity of physical phenomena presented in this paper can be used in different branches of knowledge and science. The procedure is the same in all applications. These principles and theorems can be applied both in the case when all the equations and boundary conditions of the problem are known and in the case when only the general functional relationships of the problem are known or postulated. Similarity criteria numbers can be derived in general from the original generalized theorems  $\Pi$  of dimensional analysis and model similarity of physical phenomena.

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